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Maximum Maximum of Martingales given Marginals

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Abstract

Following our previous work [11], we consider the problem of superhedging under volatility uncertainty for an investor allowed to dynamically trade the underlying asset, and statically trade European call options for all possible strikes and finitely-many maturities. The dual formulation converts this problem into a continuous-time martingale optimal transportation problem which we solve explicitly for Lookback options with nondecreasing payoff function. In particular, our methodology recovers the extensions of the Azéma-Yor solution of the Skorohod embedding problem obtained by Hobson and Klimmek [13] (under slightly different conditions), those derived by Brown, Hobson and Rogers [8], and those obtained by Madan and Yor [14].

Key words: Optimal control, volatility uncertainty, optimal transportation.

AMS 1991 subject classifications: Primary 49L25, 60J60; secondary 49L20, 35K55.

1 Introduction

The objective of this paper is to derive in explicit form the superhedging cost of a Lookback option given that the underlying asset is available for frictionless continuous-time trading, and that European options for all strikes are available for trading for a finite set of maturities. In a zero interest rate financial market, it essentially follows from the no-arbitrage condition that these trading possibilities restrict the underlying asset price process to be a martingale with given marginals.

Since a martingale can be written as a time changed Brownian motion, and the maximum of the processes is not altered by a time change, the one-marginal constraint version of this

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problem can be converted into the framework of the Skorohod embedding problem (SEP). This observation is the starting point of the seminal paper by Hobson [12] who exploited the already known optimality result of the Azéma-Yor solution to the SEP and, more importantly, provided an explicit static superhedging strategy, see also the recent work [1]. An extension to the case where two intermediate laws of the process are given was obtained by Brown, Hobson and Rogers [8]. The case where there are finitely-many intermediate laws $\mu_1, \ldots, \mu_n$, was addressed by Madan and Yor [14]. But they only provided a solution under their *increasing residual mean value property* which assumes that the corresponding barycenter functions $b_1, \ldots, b_n$ are nondecreasing, i.e. $b_1 \leq \ldots \leq b_n$. We also refer to Cox and Oblój [9] for the case of no-touch options.

The main result of this paper is to provide a solution to the multiple marginals problem under the weaker assumption that the probability measures $\mu_1, \ldots, \mu_n$ are increasing in the convex order, a property which is equivalent to the martingale property of the underlying asset price process. This answers the question left open by Madan and Yor [14] of extending the Azéma-Yor embedding to the case of n intermediate laws. We observe that, as a by-product of our result, we also recover the recent result of Hobson and Klimmek [13] under slightly different conditions.

Our approach is to exploit a duality transformation which converts our problem into a martingale transportation problem: maximize the expected coupling defined by the payoff so as to transport the Dirac measure along the given distributions $\mu_1, \ldots, \mu_n$ by means of a continuous-time process restricted to be a martingale. This approach was simultaneously suggested by [4] in the discrete-time case, and [11] in continuous-time. In contrast with the SEP approach, the martingale transportation approach is a systematic methodology to address the present problem. See Bonnans and Tan [6] for a numerical approximation in the context of variance options, and Tan and Touzi [17] for a general version of the optimal transportation problem under controlled dynamics.

Loosely speaking, the dual martingale transportation problem consists in a standard penalization of the marginal distribution constraint. An important financial interpretation is that the corresponding Lagrange multiplier represents the optimal static position in Vanilla options so as to reduce the risk induced by the derivative security. Our explicit solution also provides this Lagrange multiplier in explicit form.

The paper is organized as follows. Section 2, provides the precise mathematical formulation of the problem, and states the main duality result whose proof is reported in the Appendix. In Section 3, we prepare for our main result by solving the one-marginal problem for Lookback options with payoff depending on the underlying asset price and the corresponding running maximum at the maturity. Finally, Section 4 contains our main results in the context where the martingale is constrained by finitely-many intermediate marginals.

2 Robust superhedging of Lookback options

2.1 Modeling the volatility uncertainty

Throughout this paper, we shall consider the one-dimensional case, although all results of the present section are valid in the multi-dimensional case.

Let $\Omega := \{\omega \in C([0,T],\mathbb{R}^1) : \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\le t\le T}|\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$ the filtration generated by B. Throughout the paper, X_0 is some given initial value in $\mathbb{R}$, and we denote

$$X_t := X_0 + B_t \quad \text{for} \quad t \in [0,T].$$

For all $\mathbb{F}-$progressively measurable process σ with values in $\mathbb{R}^+$ and satisfying $\int_0^T \sigma_s^2 ds < \infty$, $\mathbb{P}_0-$a.s., we define the probability measures on $(\Omega,\mathcal{F})$:

$$\mathbb{P}^\sigma := \mathbb{P}_0 \circ (X^\sigma)^{-1} \quad \text{where} \quad X_t^\sigma := X_0 + \int_0^t \sigma_r dB_r, \ t \in [0,T], \ \mathbb{P}_0 - \text{a.s.}$$

Then X is a $\mathbb{P}^\sigma-$local martingale. Following [16], we denote by $\overline{\mathcal{P}}_S$ the collection of all such probability measures on $(\Omega,\mathcal{F})$. The quadratic variation process $\langle X\rangle = \langle B\rangle$ is universally defined under any $\mathbb{P} \in \overline{\mathcal{P}}_S$, and takes values in the set of all nondecreasing continuous functions with $\langle B\rangle_0 = 0$. Moreover, for all $\mathbb{P} \in \overline{\mathcal{P}}_S$, the quadratic variation $\langle B\rangle$ is absolutely continuous with respect to the Lebesgue measure.

Following the previous literature on quasi-sure stochastic analysis started by Denis and Martini [10], we shall use the following abuse of terminology.

Definition 2.1 *For a subset $\mathcal{P}_0 \subset \mathcal{P}_S$, we say that a property holds $\mathcal{P}_0-$quasi-surely (q.s.) if it holds $\mathbb{P}-$a.s. for every $\mathbb{P} \in \mathcal{P}_0$.*

2.2 Robust super-hedging by trading the underlying

Let n be some positive integer, $0 = t_0 < \ldots < t_n = T$ be some partition of the interval $[0,T]$, and consider the Lookback option defined by the payoff at maturity t_n:

$$\xi := G\big(X_{t_1},\ldots,X_{t_n},X_{t_n}^*\big) \quad \text{where} \quad X_t^* := \max_{r\le t} X_r$$

is the running maximum of the coordinate process.

The chief goal of this paper is to analyze the robust superhedging cost of the Lookback ξ. Since the coordinate process stands for the price process of an underlying security, we shall focus on the subset $\mathcal{P}_\infty$ of $\overline{\mathcal{P}}_S$ consisting of all measures $\mathbb{P}$ such that

$$\begin{aligned} X \ \text{ is a } \ \mathbb{P} - \text{uniformly integrable martingale,} \\ \text{and } \ X_{t_n}^* \in \mathbb{L}^1(\mathbb{P}). \end{aligned} \tag{2.1}$$

For all $\mathbb{P} \in \mathcal{P}_\infty$, we denote by $\mathbb{H}^0(\mathbb{P})$ the collection of all $(\mathbb{P}, \mathbb{F})-$progressively measurable processes. To define the investor's wealth process in the present uncertain volatility context, we introduce the set:

$$\hat{\mathbb{H}}^2_{\text{loc}} := \cap_{\mathbb{P}\in\mathcal{P}^+_\infty} \mathbb{H}^2_{\text{loc}}(\mathbb{P}) \ \text{ where } \ \mathbb{H}^2_{\text{loc}}(\mathbb{P}) := \left\{ H \in \mathbb{H}^0(\mathbb{P}) : \int_0^T |H_t|^2 d\langle B\rangle_t < \infty, \ \mathbb{P} - \text{a.s.} \right\} \quad .$$

Under the self-financing condition, any $H \in \hat{\mathbb{H}}^2_{\text{loc}}$ induces the portfolio value process

$$Y^H_t := Y_0 + \int_0^t H_s \cdot dB_s, \quad t \in [0, T]. \tag{2.2}$$

This stochastic integral is well-defined $\mathbb{P}-$a.s. for every $\mathbb{P} \in \mathcal{P}_\infty$, and should be rather denoted $Y_t^{H^{\mathbb{P}}}$ to emphasize its dependence on $\mathbb{P}$. Finally, in order to avoid possible arbitrage opportunities which may be induced by doubling strategies, we define the set of admissible strategies $\mathcal{H}$ by

$$\mathcal{H} \ := \ \left\{ H \in \hat{\mathbb{H}}^2_{\text{loc}} : \text{ for all } \mathbb{P} \in \mathcal{P}_\infty, \ Y^H \geq M^{\mathbb{P}} \ \mathbb{P} - \text{a.s. for some } \mathbb{P}-\text{martingale } M^{\mathbb{P}} \right\},$$

where the martingale $M^{\mathbb{P}}$ may depend on $\mathbb{P}$ and the portfolio strategy H. Then, it follows from (2.2) that

$$Y^H \quad \text{is a } \mathbb{P}-\text{local martingale and } \mathbb{P}-\text{supermartingale, for all} \quad H \in \mathcal{H}, \ \mathbb{P} \in \mathcal{P}_\infty. \tag{2.3}$$

The robust superhedging problem of this derivative security is defined by:

$$U^0(\xi) \ := \ \inf \left\{ Y_0 : \ Y^H_{t_n} \geq \xi, \ \mathcal{P}_\infty - \text{q.s. for some } H \in \mathcal{H} \right\}. \tag{2.4}$$

The subsequent result provides a dual formulation of the robust superhedging problem similar to [16] and [11], but under weaker conditions on ξ. The proof uses strongly the particular structure of the derivative security ξ, and follows the lines of the construction of the $G-$expectation in Peng [15].

Theorem 2.1 *Assume that $\xi^+ \in \mathbb{L}^1(\mathbb{P})$ for all $\mathbb{P} \in \mathcal{P}_\infty$. Then*

$$U^0(\xi) \ = \ \sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi],$$

and existence holds for the robust superhedging problem $U^0(\xi)$, whenever $U^0(\xi) < \infty$.

Proof See the Appendix Section 5.1. □

2.3 Robust superhedging with additional trading of Vanillas

We assume that, in addition to the continuous-time trading of the primitive securities, the investor can take static positions in European call or put options with all possible strikes and maturities $t_{i_1} < \cdots < t_{i_p}$ for some integers $p \leq n$ and $i_1 < \ldots < i_p \leq n$. The market price of the European call option with strike $K \geq 0$ and maturity t_{i_j} is denoted

$$c_{i_j}(K), \quad j = 1, \ldots, p, \quad \text{and we denote} \quad c_0(K) := (X_0 - K)^+.$$

Then, from Breeden and Litzenberger [7], the investor can identify the $t_{i_j}-$marginal distribution $\mu_{i_j} \in M(\mathbb{R})$, the collection of all probability measures on $\mathbb{R}$, of the underlying asset under the pricing measure by $\mu_{i_j}((K, \infty)) := c'_{i_j}(K+)$, the right hand-side derivative of the convex function c_{i_j} at K. In particular, the $t_{i_j}-$maturity European derivative defined by the payoff $\lambda_{i_j}(X_{t_{i_j}})$ has an un-ambiguous no-arbitrage price

$$\mu_{i_j}(\lambda_{i_j}) = \int \lambda_{i_j} d\mu_{i_j}.$$

Remark 2.1 For the purpose of the present financial application, we may restrict the measure μ to have support in $\mathbb{R}_+$. We consider however the general case $\mu \in M(\mathbb{R})$ in order to compare our results to the literature on the SEP.

As it will be made clear in our subsequent Proposition 2.1, the function λ_{i_j} is in fact a Lagrange multiplier for the constraint $X_{t_{i_j}} \sim \mu_{i_j}$, $j = 1, \ldots, p$. Of course, the martingale property of the underlying imposes the restriction that the probability measures μ_{i_j} are nondecreasing in the convex order or, equivalently, that

$$c_{i_{j-1}} \leq c_{i_j} \quad \text{for all} \quad j = 1, \ldots, p. \tag{2.5}$$

We denote $\mu := (\mu_{i_1}, \ldots, \mu_{i_p})$, $t := (t_{i_1}, \ldots, t_{i_p})$, $\lambda = (\lambda_{i_1}, \ldots, \lambda_{i_p})$,

$$\mu(\lambda) := \sum_{j=1}^{p} \mu_{i_j}(\lambda_{i_j}), \;\; \lambda(\mathbf{x}_t) := \sum_{j=1}^{p} \lambda_{i_j}(\mathbf{x}_{t_{i_j}}), \;\text{ and } G^{\lambda}(\mathbf{x}_t, m) := G(\mathbf{x}_{t_1}, \ldots, \mathbf{x}_{t_n}, m) - \lambda(\mathbf{x}) \tag{2.6}$$

for $\mathbf{x} \in C^0([0, t_n])$ and $(\mathbf{x}_0, m) \in \mathbf{\Delta}$ with

$$\mathbf{\Delta} \;\; := \;\; \big\{(x, m) \in \mathbb{R}^2 : \; m \geq x\big\}. \tag{2.7}$$

The set of Vanilla payoffs which may be used by the hedger are naturally taken in the set

$$\Lambda^{\mu}_{n} := \Big\{\lambda : \; \lambda_{i_j} \in \mathbb{L}^1(\mu_{i_j}), j = 1, \ldots, p, \;\text{ and } G^{\lambda}\big(X_t, X^*_{t_n}\big)^+ \in \cap_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{L}^1(\mathbb{P})\Big\}. \tag{2.8}$$

The no-arbitrage upper bound is defined by:

$$U_n^\mu(\xi) \;:=\; \inf\left\{Y_0 : \overline{Y}_T^{H,\lambda} \geq \xi, \mathcal{P}_\infty - \text{q.s. for some } H \in \mathcal{H} \text{ and } \lambda \in \Lambda_n^\mu\right\}, \tag{2.9}$$

where $\overline{Y}^{H,\lambda}$ denotes the portfolio value of a self-financing strategy with continuous trading H in the primitive securities, and static trading λ_i in the t_i-maturity European calls with all strikes:

$$\overline{Y}_T^{H,\lambda} \;:=\; Y_T^H - \mu(\lambda) + \lambda(X_T), \tag{2.10}$$

indicating that the investor has the possibility of buying at time 0 any derivative security with payoff $\lambda_i(X_{t_i})$ for the price $\mu_i(\lambda_i)$. Similar to [11], the next result is a direct consequence of the robust superhedging dual formulation of Theorem 2.1.

Proposition 2.1 *Assume that $\xi^+ \in \mathbb{L}^1(\mathbb{P})$ for all $\mathbb{P} \in \mathcal{P}_\infty$, and let the family of probability measures μ_i, $i = 1, \ldots, n$ be nondecreasing in the convex order, i.e.* (2.5)*. Then:*

$$U_n^\mu(\xi) \;=\; \inf_{\lambda \in \Lambda_n^\mu} \sup_{\mathbb{P} \in \mathcal{P}_\infty} \left\{\mu(\lambda) + \mathbb{E}^{\mathbb{P}}\left[G^\lambda\left(X_{t_1}, \ldots, X_{t_n}, X_{t_n}^*\right)\right]\right\}.$$

Our objective in the subsequent sections is to use the last dual formulation in order to obtain a closed form expression for the above upper bound in the following special cases

- the one-marginal case under some "monotonicity" condition of G in m,
- the multiple-marginal case

$$G(x_1, \ldots, x_n, m) = \phi(m) \quad \text{for some nondecreasing Lipschitz function} \quad \phi.$$

The one-marginal result is reported in Section 3, and has been recently established by Hobson and Klimmek [13] under slightly different assumptions; therefore it must be viewed as an alternative approach to that of [13]. In contrast, the multiple-marginal result of Section 4 is new to the literature, and generalizes previous contributions of Brown, Hobson and Rogers [8] in the two-marginal case, and Madan and Yor [14] in the multiple-marginal case under their *increasing mean residual value property.*

2.4 Optimal transportation and Skorohod embedding problem

In this short section we discuss the connection of our problem to optimal transportation theory, on one hand, and to the Skorohod embedding problem, on the other hand.

First, by formally inverting the inf-sup in the dual formulation of Proposition 2.1, we see that $U_n^\mu(\xi)$ is related to the optimization problem:

$$\sup_{\mathbb{P} \in \mathcal{P}_\infty(\mu)} \mathbb{E}^{\mathbb{P}}[\xi], \quad \text{with} \quad \mathcal{P}_\infty(\mu) := \left\{\mathbb{P} \in \mathcal{P}_\infty : \; X_{t_i} \sim \mu_i, \; 1 \leq i \leq n\right\}, \tag{2.11}$$

which falls in the recently introduced class of optimal transportation problems under controlled stochastic dynamics, see [4, 11, 17]. In words, the above problem consists in maximizing the expected transportation cost of the Dirac measure $\delta_{\{X_0\}}$ along the given marginals $\mu_1, \ldots, \mu_n$ with transportation scheme constrained to a specific subclass of martingales. The cost of transportation in our context is defined by the payoff function of the Lookback option $G(\mathbf{x}, m)$.

We observe however that whether or not the value function in (2.11) coincides with our problem $U_n^\mu(\xi)$ is an open problem, see however [4] in the discrete-time context.

Next, by the Dambis-Dubins-Schwartz time change theorem, we may re-write the problem (2.11) as a multiple stopping problem (see Proposition 3.1 in [11]):

$$\sup_{(\tau_1,\ldots,\tau_n)\in\mathcal{T}(\mu)} \mathbb{E}^{\mathbb{P}_0}\big[G\big(X_{\tau_1}, \ldots, X_{\tau_n}, X^*_{\tau_n}\big)\big], \tag{2.12}$$

where the $\mathcal{T}(\mu)$ is a convenient set of ordered stopping times with $X_{\tau_i} \sim_{\mathbb{P}_0} \mu_i$ for all $i = 1, \ldots, n$.

The problem (2.12) is related to the so-called Skorohod embedding problem (SEP) of finding stopping times $\tau_1, \ldots, \tau_n$ such that the $\mathbb{P}_0-$distribution of X_{τ_i} is μ_i for all $i = 1, \ldots, n$. Here, the formulation (2.12) is directly searching for a solution of the SEP which maximizes the criterion defined by the coupling $G(x, m)$. The case $G(x, m) = \phi(m)$ for some nondecreasing function ϕ is solved by the so-called Azéma-Yor embedding [2, 3, 12], see also our work [11] which recovers this result by our approach. The case $G(x, m)$ was considered recently by Hobson and Klimmek [13], where the optimality of the Azéma-Yor solution of the SEP is shown to be valid under convenient conditions on the function G. This case is also solved in Section 3 of the present paper with our approach, leading to the same results than [13] but under slightly different conditions.

The case $n = 2$ with $G(x_1, x_2, m) = \phi(m)$ for some nonincreasing function ϕ was solved in Brown, Hobson and Rogers [8]. Their results are crucial for our approach in the following sense. Our approach leads naturally to the minimization problem which was the starting point of [8], and we use their result that the solution of the finite-dimensional optimization problem indeed solves the corresponding SEP.

Finally, the general multiple-marginals case was considered by Madan and Yor [14] under their *increasing mean residual value property*. As a consequence their result does not encompass the two-marginal result of [8]. The main contribution of this paper is to fill this gap in the literature, thus solving completely the problem which was left open in [14].

3 The one marginal problem

For an inherited maximum $M_0 \geq X_0$, we introduce the process:

$$M_t := M_0 \vee X_t^* \quad \text{for} \quad t \geq 0.$$

The process (X, M) takes values in the state space $\mathbf{\Delta}$ introduced in (2.7). Our interest in this section is on the upper bound on the price of the Lookback option defined by the payoff

$$\xi \;=\; g(X_T, X_T^*) \tag{3.1}$$

where $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies some conditions to be specified later.

For a function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$, we denote $g^\lambda := g - \lambda$, and we recall the notation

$$\Lambda^\mu = \left\{\lambda \in \mathbb{L}^1(\mu) : \; g^\lambda(X_T, M_T)^+ \in \cap_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{L}^1(\mathbb{P})\right\} \tag{3.2}$$

for all probability measure $\mu \in M(\mathbb{R})$. Similar to Proposition 3.1 in [11], it follows from the Dambis-Dubins-Schwartz time change theorem that the model-free upper bound can be converted into:

$$U^\mu(\xi) := \inf_{\lambda\in\Lambda^\mu} \sup_{\tau\in\mathcal{T}_\infty} \left\{\mu(\lambda) + J(\lambda, \tau)\right\} \quad \text{where} \quad J(\lambda, \tau) := \mathbb{E}^{\mathbb{P}_0}\left[g^\lambda(X_\tau, X_\tau^*)\right]. \tag{3.3}$$

Then for every fixed multiplier $\lambda \in \Lambda^\mu$, we are facing the infinite horizon optimal stopping problem

$$u^\lambda(x, m) := \sup_{\tau\in\mathcal{T}_\infty} \mathbb{E}^{\mathbb{P}_0}_{x,m}\left[g^\lambda(X_\tau, M_\tau)\right], \quad (x, m) \in \mathbf{\Delta}, \tag{3.4}$$

where $\mathbb{E}^{\mathbb{P}_0}_{x,m}$ denotes the conditional expectation operator $\mathbb{E}^{\mathbb{P}_0}[.|(X_0, M_0) = (x, m)]$, and $\mathcal{T}_\infty$ is the collection of all stopping times τ such that

$$\begin{gathered}\{X_{t\wedge\tau}, t \geq 0\} \text{ is a } \mathbb{P}_0\text{−uniformly integrable martingale} \\ \text{and } \; \mathbb{E}^{\mathbb{P}_0}\left[X_\tau^*\right] < \infty.\end{gathered} \tag{3.5}$$

Finally, the set Λ^μ of (3.2) translates in the present context to:

$$\Lambda^\mu \;=\; \left\{\lambda \in \mathbb{L}^1(\mu) : \; g^\lambda(X_\tau, M_\tau)^+ \in \mathbb{L}^1(\mathbb{P}_0) \text{ for all } \tau \in \mathcal{T}_\infty\right\}. \tag{3.6}$$

Remark 3.1 By the Doob's $\mathbb{L}^1$ inequality, the condition $\mathbb{E}^{\mathbb{P}_0}\left[X_\tau^*\right]$ is equivalent to $\mathbb{E}^{\mathbb{P}_0}\left[X_\tau(\ln X_\tau)^+\right] < \infty$.

3.1 A first inequality

In this subsection, we isolate the key step for the one-marginal problem. This step will also be used again in the derivation of the multiple marginals bounds. For the reader's convenience we collect the conditions on the payoff function $g(x, m)$ required here.

Assumption A *Function $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is C^1 in (x, m), Lipschitz in m uniformly in x, and g_{xx} exists as a measure.*

Assumption B *The function $x \longmapsto \frac{g_m(x,m)}{m-x}$ is nondecreasing.*

The dynamic programming equation corresponding to the optimal stopping problem u^λ defined in (3.4) is:

$$\begin{aligned} &\min\left\{u - g^\lambda, -u_{xx}\right\} = 0 \;\text{ for }\; (x, m) \in \boldsymbol{\Delta} \\ &u_m(m, m) = 0 \;\text{ for }\; m \geq 0. \end{aligned} \tag{3.7}$$

It is then natural to introduce a candidate solution for the dynamic programming equation defined by a free boundary $\{x = \psi(m)\}$, for some convenient function ψ:

$$v^\psi(x, m) = g^\lambda(x \wedge \psi(m), m) + (x - \psi(m))^+ g_x^\lambda(\psi(m), m) \tag{3.8}$$

$$= g^\lambda(x, m) - \int_{\psi(m)}^{x \vee \psi(m)} (x - \xi) g_{xx}^\lambda(\xi, m) d\xi, \quad 0 \leq x \leq m, \tag{3.9}$$

i.e. $v^\psi(., m)$ coincides with the obstacle g^λ before the exercise boundary $\psi(m)$, and satisfies $v_{xx}^\psi(., m) = 0$ in the continuation region $[\psi(m), m]$. However, the candidate solution needs to satisfy more conditions. Namely $v^\psi(., m)$ must be above the obstacle, concave in x on $(-\infty, m]$, and it needs to satisfy the Neumann condition in (3.7).

For this reason, our strategy of proof consists in first restricting the minimization in (3.3) to those multipliers λ in the set:

$$\hat{\Lambda}^\mu := \left\{\lambda \in \Lambda^\mu : v^\psi \text{ concave in } x \text{ and } v^\psi \geq g^\lambda \text{ for some } \psi \in \Psi^\lambda\right\}, \tag{3.10}$$

where the set Ψ^λ is defined in (3.13) below so that our candidate solution v^ψ satisfies the Neumann condition in (3.7). Namely, by formal differentiation of v^ψ, the Neumann condition reduces to the ordinary differential equation (ODE):

$$-\psi' g_{xx}^\lambda(\psi, m) = \gamma(\psi, m) \quad \text{where} \quad \gamma(x, m) := (m - x)\frac{\partial}{\partial x}\left\{\frac{g_m(x, m)}{m - x}\right\} \tag{3.11}$$

exists a.e. in view of Assumption B. Similar to [11], we need for technical reasons to consider this ODE in the relaxed sense. Since g^λ is concave in x on $(-\infty, \psi(m)]$, the partial second derivative g_{xx}^λ is well-defined as a measure on $\mathbb{R}$. We then introduce the weak formulation

of the ODE (3.11):

$$\begin{aligned} &\psi(m) < m \;\text{ for all }\; m\in\mathbb{R}, \\ \text{and}\quad &-\int_{\psi(E)} g^\lambda_{xx}(.,\psi^{-1})(d\xi) = \int_E \gamma(\psi,.)(dm) \;\text{ for all }\; E\in\mathcal{B}(\mathbb{R}), \end{aligned} \tag{3.12}$$

where ψ is chosen in its right-continuous version, and is nondecreasing by the concavity of g^λ and the nonnegativity of γ implied by Assumption B. We introduce the collection of all relaxed solutions of (3.11):

$$\Psi^\lambda \;:=\; \big\{\psi:\mathbb{R}\to\mathbb{R} \text{ right-continuous and satisfies (3.12)}\big\}. \tag{3.13}$$

Notice that the ODE (3.11), which motivates the relaxation (3.12), does not characterize the free boundary ψ as it is not complemented by any boundary condition.

Remark 3.2 For later use, we observe that (3.12) implies by direct integration that

the function $\;x\longmapsto \lambda(x) - \int_{\psi(X_0)}^x\int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi),\xi)}{\xi-\psi(\xi)}d\xi dy - \int_{\psi(X_0)}^x g_x(\xi,\psi^{-1}(\xi))d\xi\;$ is affine.

Proposition 3.1 *Let Assumptions A and B hold true. Then:*

$$u^\lambda \le v^\psi \quad \textit{for any} \quad \lambda\in\hat\Lambda^\mu \quad \textit{and} \quad \psi\in\Psi^\lambda.$$

Proof Assumption B guarantees that the function ψ that we will be manipulating is nondecreasing and has a well defined right-continuous inverse. We proceed in three steps.
1. We first prove that v^ψ is differentiable in m on the diagonal with

$$v^\psi_m(m,m) = 0 \quad \text{for all} \quad m\in\mathbb{R}. \tag{3.14}$$

Indeed, since $\psi\in\Psi_\lambda$, it follows from Remark 3.2 that

$$\lambda(x) = \alpha_0 + \alpha_1 x + \int_{\psi(X_0)}^x\int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi),\xi)}{\xi-\psi(\xi)}d\xi dy + \int_{\psi(X_0)}^x g_x(\xi,\psi^{-1}(\xi))d\xi$$

for some scalar constants α_0,α_1. Plugging this expression into (3.8), we see that for $\psi(m)\le x\le m$:

$$\begin{aligned} v^\psi(x,m) &= g(\psi(m),m) - \Big(\alpha_1 + \int_{X_0}^m \frac{g_m(\psi(\xi),\xi)}{\xi-\psi(\xi)}d\xi\Big)\big(x-\psi(m)\big) \\ &\quad -\Big(\alpha_0+\alpha_1\psi(m) + \int_{\psi(X_0)}^{\psi(m)}\int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi),\xi)}{\xi-\psi(\xi)}d\xi dy + \int_{\psi(X_0)}^{\psi(m)} g_x(\xi,\psi^{-1}(\xi))d\xi\Big) \\ &= g\big(\psi(m),m\big) - \alpha_0 - \alpha_1 x - \int_{X_0}^m g_m\big(\psi(\xi),\xi\big)\frac{x-\psi(\xi)}{\xi-\psi(\xi)}d\xi - \int_{\psi(X_0)}^{\psi(m)} g_x\big(\xi,\psi^{-1}(\xi)\big)d\xi. \end{aligned}$$

Since g is C^1, (3.14) follows by direct differentiation with respect to m.

2. Let $\tau \in \mathcal{T}_\infty$ be arbitrary. Clearly, it is sufficient to restrict attention to those $\tau \in \mathcal{T}_\infty$ such that $g^\lambda(X_\tau, M_\tau) \in \mathbb{L}^1(\mathbb{P}_0)$.

Define the sequence of stopping times $\tau_n := \tau \wedge \inf\{t > 0 : |X_t - x| > n\}$. Since v^ψ is concave, it follows from the Itô-Tanaka formula that:

$$v^\psi(x,m) \;\; \geq \;\; v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) dB_t - \int_0^{\tau_n} v_m^\psi(X_t, M_t) dM_t$$

Notice that $(M_t - X_t)dM_t = 0$. Then since $v_m^\psi(m,m) = 0$, it follows that $v_m^\psi(X_t, M_t)dM_t = v_m^\psi(M_t, M_t)dM_t = 0$, and therefore:

$$v^\psi(x,m) \;\; \geq \;\; v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) dX_t.$$

Taking expectations in the last inequality, we see that:

$$v^\psi(x,m) \geq \mathbb{E}_{x,m}^{\mathbb{P}_0}\left[v^\psi(X_{\tau_n}, M_{\tau_n})\right]. \tag{3.15}$$

3. By the Lipschitz property of g in m uniformly in x (Assumption A):

$$\left|g^\lambda\big(X_\tau, M_{\tau_n}\big)\right| \;\; \leq \;\; \left|g^\lambda\big(X_\tau, M_\tau\big)\right| + CM_\tau$$

for some constant C. Since $g^\lambda\big(X_\tau, M_\tau\big) \in \mathbb{L}^1(\mathbb{P}_0)$ and $M_\tau \in \mathbb{L}^1(\mathbb{P}_0)$, by the definition of $\mathcal{T}_\infty$, this shows that $g^\lambda(X_\tau, M_{\tau_n}) \in \mathbb{L}^1(\mathbb{P}_0)$. We next deduce from the concavity of v^ψ in x that:

$$v^\psi\big(X_{\tau_n}, M_{\tau_n}\big) + v_x^\psi\big(X_{\tau_n}, M_{\tau_n}\big)(B_\tau - B_{\tau_n}) \;\; \geq \;\; v^\psi\big(X_\tau, M_{\tau_n}\big).$$

Since $(B_{t\wedge\tau})_{t\geq 0}$ is a uniformly integrable martingale, this provides:

$$v^\psi\big(X_{\tau_n}, M_{\tau_n}\big) \;\; \geq \;\; \mathbb{E}^{\mathbb{P}_0}\Big[v^\psi\big(X_\tau, M_{\tau_n}\big)\big|\mathcal{F}_{\tau_n}\Big] \;\geq\; \mathbb{E}^{\mathbb{P}_0}\Big[g^\lambda\big(X_\tau, M_{\tau_n}\big)\big|\mathcal{F}_{\tau_n}\Big], \tag{3.16}$$

where the last inequality follows from the fact that v^ψ is above the obstacle g^λ. Then it follows from (3.15) together with the tower property of conditional expectations that $v(x,m) \geq \mathbb{E}_{x,m}^{\mathbb{P}_0}\big[g^\lambda(X_\tau, M_{\tau_n})\big]$. Using again Assumption A, we then see that:

$$v^\psi(x,m) \;\; \geq \;\; \mathbb{E}_{x,m}^{\mathbb{P}_0}\left[g^\lambda(X_\tau, M_\tau) - C(M_\tau - M_{\tau_n})\right] \nearrow \mathbb{E}_{x,m}^{\mathbb{P}_0}\left[g^\lambda(X_\tau, M_\tau)\right],$$

by the monotone convergence theorem. By the arbitrariness of $\tau \in \mathcal{T}_\infty$, this implies that $v^\psi \geq u^\lambda$. □

Remark 3.3 In the special case $g(x,m) = \phi(m)$ for some C^1 nondecreasing function ϕ, the Lipschitz property in Assumption 3.2 (A) can be dropped by using the monotone convergence theorem in the passage to the limit after equation (3.16), see [11].

3.2 The Azema-Yor embedding solves the one-marginal problem

The endpoints of the support of the distribution μ are denoted by:

$$\ell^\mu := \sup\big\{x : \mu\big([x,\infty)\big) = 1\big\} \quad \text{and} \quad r^\mu := \inf\big\{x : \mu\big((x,\infty)\big) = 0\big\}$$

We introduce the so-called barycenter function:

$$b(x) := \frac{\int_{[x,\infty)} y\mu(dy)}{\mu\big([x,\infty)\big)}\, \mathbf{1}_{\{x<r^\mu\}} + x\, \mathbf{1}_{\{x\geq r^\mu\}} \quad x \geq 0. \tag{3.17}$$

The Azéma-Yor [2, 3] solution of the *Skorohod Embedding Problem* is:

$$\tau^* \;\; := \;\; \inf\big\{t > 0 : \; X_t^* \geq b(X_t)\big\}. \tag{3.18}$$

In this subsection, we need the following additional condition on the payoff function $g(x,m)$.

Assumption C *Either one of the following conditions hold:*

(C1) *g is Lipschitz in x uniformly in m, and*

$$g_{xx}(dx,m) - g_{xx}(dx,b(x)) \leq \gamma(x,b(x))b(dx) \quad \textit{whenever} \quad b(x) \leq m.$$

(C2) *$g_x(\ell^\mu, X_0)$, $g_x(b^{-1}(r^\mu), r^\mu) \in \mathbb{R}$, and g_m is concave in x.*

Define the function

$$\lambda^*(x) := \int_{\ell^\mu}^x \int_{\ell^\mu}^y g_m\big(\xi, b(\xi)\big)\frac{\mu(d\xi)}{\mu([\xi,\infty))}dy + \int_{\ell^\mu}^x g_x(\xi, b(\xi))d\xi; \quad x \in (-\infty, r^\mu), \tag{3.19}$$

which appears naturally by plugging $\psi^* := b^{-1}$ in the ODE (3.12) that will be introduced below. We next observe that $\lambda^* \in \mathbb{L}^1(\mu)$. Indeed, following Step 1 of the proof of Lemma 3.2 in [11], this is equivalent to the integrability of $c(.)$ with respect to the measure $(\lambda^*)''$, and it follows from the ODE (3.12) that

$$\begin{aligned}
\int c(x)(\lambda^*)''(dx) \;\; &= \;\; \int_{X_0}^\infty c(\psi^*(m))\big(\gamma(\psi^*(m),m)dm + g_{xx}(\psi^*(m),m)d\psi^*(m)\big) \\
&= \;\; \int_{X_0}^\infty c(\psi^*(m))\Big(\frac{g_m(\psi^*(m),m)}{m-\psi^*(m)}dm + dg_x(\psi^*(m),m)\Big) \\
&= \;\; \int_{X_0}^\infty \frac{c(\psi^*(m))}{m-\psi^*(m)}g_m(\psi^*(m),m)dm + \int_{X_0}^\infty c(\psi^*(m))dg_x(\psi^*(m),m)
\end{aligned}$$

Since $c(\psi^*(\infty)) = 0$ and $c(\ell^\mu) = X_0$, the second integral is well-defined and finite either by
- the boundedness of g_x in Assumption C1,

- or by the finiteness of $g_x(\ell^\mu, X_0)$ and $g_x(\psi(r^\mu), r^\mu)$ in Assumption C2 (which implies that $m \longmapsto g_x(\psi^*(m), m)$ is bounded).

As for the first integral, it follows from the boundedness of g_m in Assumption A that $\int_{X_0}^\infty \frac{c(\psi^*(m))}{m-\psi^*(m)} |g_m(\psi^*(m), m)| dm \le |g_m|_\infty \int_{X_0}^\infty \frac{c(\psi^*(m))}{m-\psi^*(m)} dm < \infty$. Hence $\lambda^* \in \mathbb{L}^1(\mu)$.

The following result has been derived recently by [13] under slightly different conditions than those in Assumption 3.2. Our objective is to derive it directly from our dual formulation.

Theorem 3.1 *Let ξ be given by* (3.1) *for some payoff function g satisfying Assumptions A, B, and C. Then, (λ^*, τ^*) is a solution of the problem U^μ in* (3.3), *and for any $\mu \in M(\mathbb{R})$ with $\mathbb{E}^{\mathbb{P}_0}[g(X_{\tau^*}, X^*_{\tau^*})] < \infty$, we have:*

$$U^\mu(\xi) \;=\; J(\lambda^*, \tau^*) \;=\; \mathbb{E}^{\mathbb{P}_0}[g(X_{\tau^*}, X^*_{\tau^*})].$$

The remaining part of this section is dedicated to the proof of this result. We notice that none of the arguments of this proof will not be needed for the multi-marginals case.

Our starting point is the result of Proposition 3.1 which provides an upper bound for the value function $U^\mu(\xi)$ for every choice of a multiplier $\lambda \in \hat{\Lambda}^\mu$ and a corresponding solution $\psi \in \Psi^\lambda$ of the ODE (3.12):

$$U^\mu(\xi) \le \mu(\lambda) + v^\psi(X_0, X_0) \quad \text{for all} \quad \lambda \in \hat{\Lambda}^\mu \;\; \text{and} \;\; \psi \in \Psi^\lambda. \tag{3.20}$$

Alternatively, for any choice of a nondecreasing function ψ with $\psi(m) < m$ for all $m \in \mathbb{R}$, we may define a corresponding multiplier function λ by (3.12), or equivalently by (3.11), in the distribution sense. Then $\psi \in \Psi^\lambda$. If in addition v^ψ is concave in x and above the corresponding obstacle g^λ, then $\lambda \in \hat{\Lambda}^\mu$ and we may conclude by Proposition 3.1 that $U^\mu(\xi) \le v^\psi$. The next result exhibits this bound for the choice $\psi = b^{-1}$, the right-continuous inverse of the barycenter function.

Proposition 3.2 *Let ξ be given by* (3.1). *Then, under Assumptions A, B and C, we have:*

$$U^\mu(\xi) \;\le\; \mu(\lambda^*) + J(\lambda^*, \tau^*) = \mathbb{E}^{\mathbb{P}_0}[g(X_{\tau^*}, X^*_{\tau^*})].$$

Proof It is immediately checked that $\psi^* := b^{-1} \in \Psi^{\lambda^*}$. Then, in view of the previous discussion, the required inequality follows from Proposition 3.1 once we prove that v^{ψ^*} is concave, that $v^{\psi^*} \ge g^{\lambda^*}$, and that $\lambda^* \in \hat{\Lambda}^\mu$.

1. We first verify that v^{ψ^*} is concave. By direct computation using the expression of λ^* in (3.19) together with the identity

$$\frac{b(dx)}{b(x)-x} \;=\; \frac{\mu(dx)}{\mu([x,\infty))},$$

we see that

$$g^{\lambda^*}_{xx}(x,m) \;=\; g_{xx}(x,m) - g_{xx}\big(x,b(x)\big) - \gamma\big(x,b(x)\big)b'(x) \tag{3.21}$$

in the distribution sense. By Assumption C1, it follows that $x \longmapsto g^{\lambda^*}(x,m)$ is concave on $(-\infty,\psi^*(m)]$. We reach the same conclusion using Assumption C2 as $\gamma\big(x,b(x)\big)b'(x) \geq 0$, a.e. by Assumption B.

Since $v^\psi(.,m)$ is linear on $[\psi^*(m),m]$ and C^1 across the boundary $\psi^*(m)$, this proves that v^ψ is concave.

2. We next check that $v^{\psi^*} \geq g^{\lambda^*}$. Since equality holds on $(-\infty,\psi^*(m)]$, we only compute for $x \in [\psi^*(m),m]$ that:

$$\begin{aligned}
\big(v^{\psi^*} - g^{\lambda^*}\big)(x,m) &= \int_{\psi^*(m)}^{x} \big(g^{\lambda^*}_x(\psi^*(m),m) - g^{\lambda^*}_x(\xi,m)\big)d\xi \\
&= -\int_{\psi^*(m)}^{x}\int_{\psi^*(m)}^{\xi} g^{\lambda^*}_{xx}(y,m)dyd\xi.
\end{aligned}$$

By (3.21), this provides:

$$\begin{aligned}
\big(v^{\psi^*} - g^{\lambda^*}\big)(x,m) &= -\int_{\psi^*(m)}^{x}\Big(g_x(\xi,m) - g_x\big(\xi,b(\xi)\big) - \int_{\psi^*(m)}^{\xi}\frac{g_m(y,b(y))}{b(y)-y}b(dy)\Big)d\xi \\
&= \int_{\psi^*(m)}^{x}\int_{\psi^*(m)}^{\xi}\Big(g_{xm}\big(\xi,b(y)\big) + \frac{g_m\big(y,b(y)\big)}{b(y)-y}\Big)b(dy)d\xi \\
&= \int_{\psi^*(m)}^{x}\Big(\int_{y}^{x} g_{xm}(\xi,b(y))d\xi + \frac{g_m\big(y,b(y)\big)}{b(y)-y}\Big)b(dy) \\
&= \int_{\psi^*(m)}^{x}(b(y)-x)\Big(\frac{g_m(x,b(y))}{b(y)-x} - \frac{g_m(y,b(y))}{b(y)-y}\Big)b(dy) \;\geq\; 0,
\end{aligned}$$

where the last inequality follows the nondecrease of the functions b and $x \longmapsto g_m(x,m)/(m-x)$ (Assumption B), together with the fact that $b(y) \geq x$ for $\psi(m) \leq y \leq x \leq m$.

3. In this step, we verify that

$$g^{\lambda^*}(X_\tau,M_\tau)^+ \in \mathbb{L}^1(\mathbb{P}_0) \quad \text{for all} \quad \tau \in \mathcal{T}^\infty_+. \tag{3.22}$$

Indeed, since $g_x(y,b(y)) = g_x(y,X_0) + \int_{\ell^\mu}^{y} g_{xm}(y,b(\xi))b(d\xi)$, we compute from the expression of λ^* that:

$$\begin{aligned}
g^{\lambda^*}(x,m) &= g(x,m) - \int_{\ell^\mu}^{x} g_x(y,X_0)dy - \int_{\ell^\mu}^{x}\int_{\ell^\mu}^{y}\Big(\frac{g_m(\xi,b(\xi))}{b(\xi)-\xi} + g_{xm}(y,b(\xi))\Big)b(d\xi)dy \\
&= g(x,m) - g(x,X_0) + g(\ell^\mu,X_0) - \int_{\ell^\mu}^{x}\int_{\ell^\mu}^{y}\Big(\frac{g_m(\xi,b(\xi))}{b(\xi)-\xi} + g_{xm}(y,b(\xi))\Big)b(d\xi)dy
\end{aligned} \tag{3.23}$$

By direct integration by parts, we get

$$g^{\lambda^*}(x,m) \;=\; g(x,m) - xg_x(l^\mu, X_0) - \int_{\ell^\mu}^x \int_{\ell^\mu}^y \big(\gamma(\xi, b(\xi))b(d\xi) + g_{xx}(\xi, b(\xi))d\xi\big)dy$$

We now split the proof in two independent parts which use either C1 or C2 of Assumption C.

a. By using Assumption C1 with $m = b(y)$, we deduce that:

$$\begin{aligned} g^{\lambda^*}(x,m) &\leq g(x,m) - xg_x(l^\mu, X_0) - \int_{\ell^\mu}^x \int_{\ell^\mu}^y g_{xx}(\xi, b(y))d\xi \\ &\leq g(x,m) - xg_x(\ell^\mu, X_0) - \int_{\ell^\mu}^x \left(g_x(y, b(y)) - g_x(\ell^\mu, b(y))\right)dy \end{aligned}$$

By using Assumption C1, we get that $\int_{\ell^\mu}^x \left(g_x(y,b(y)) - g_x(\ell^\mu, b(y))\right)dy \in \mathbb{L}^1(\mathbb{P}_0)$ and therefore $g^{\lambda^*}(x,m)^+ \in \mathbb{L}^1(\mathbb{P}_0)$.

b. By the condition that g_m is concave in x in Assumption C2, we have $g_{xm}(y, b(\xi)) \leq g_{xm}(\xi, b(\xi))$ for $\xi \leq y$. Then from Equation (3.23),

$$\begin{aligned} g^{\lambda^*}(x,m) &\leq g(x,m) - g(x, X_0) + g(\ell^\mu, X_0) - \int_{\ell^\mu}^x \int_{\ell^\mu}^y \gamma(\xi, b(\xi))b(d\xi)dy \\ &\leq C|m - X_0| + g(\ell^\mu, X_0), \end{aligned}$$

where the last inequality follows from the Lipschitz property of g in m (Assumption A) and the non-negativity of the function γ (Assumption B). By the definition of $\mathcal{T}_+^\infty$, this estimate implies (3.22). □

Before turning to the proof of the converse inequality to that of Proposition 3.2, we provide a formal justification that the function b^{-1} appears naturally if one searches for the best upper bound in (3.20).

<u>*Step 1*</u>: using the expression (3.9) of v^ψ, we directly compute that

$$\begin{aligned} \mu(\lambda) + u^\lambda(X_0, X_0) &= \mu\big(g(.,X_0)\big) + \mu\big(g^\lambda(.,X_0)\big) - \int_{\psi(X_0)}^{X_0} g^\lambda_{xx}(\xi, X_0)(X_0 - \xi)d\xi \\ &= \mu\big(g(.,X_0)\big) + \int g^\lambda_{xx}(\xi, X_0)\big(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \leq \psi(X_0)\}}\big)d\xi \\ &= \mu\big(g(.,X_0)\big) + \int g^\lambda_{xx}\big(\xi, \psi^{-1}(\xi)\big)\big(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \leq \psi(X_0)\}}\big)d\xi \\ &\quad + \int \big(g_{xx}(\xi, X_0) - g_{xx}\big(\xi, \psi^{-1}(\xi)\big)\big)\big(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \leq \psi(X_0)\}}\big)d\xi, \end{aligned}$$

where the second equality follows from two integrations by parts together with the fact that $\int x\mu(dx) = X_0$, see Step 1 of the proof of Lemma 3.2 in [11]. Then, by using the ODE (3.12) satisfied by ψ to change variables in the last integral, we see that:

$$\mu(\lambda) + u^\lambda(X_0, X_0) = \mu\big(g(.,X_0)\big) + \int \big\{ -\gamma\big(\psi(m), m\big) + G\big(\psi(m), m\big)\psi'(m)\big\}\delta\big(\psi(m), m\big)dm,$$

where we denoted:

$$\delta(x,m) := c(x) - c_0(x)\mathbf{1}_{\{m\le X_0\}}, \quad c_0(x) := (X_0-x)^+,$$
$$\text{and} \;\; G(x,m) := g_{xx}(x,X_0) - g_{xx}(x,m).$$

<u>*Step 2*</u>: The expression of $\mu(\lambda)+v^\psi$ derived in the previous step only involves the function $\psi\in\Psi^\lambda$. Forgetting about all constraints on ψ, we treat our minimization problem as a standard problem of calculus of variations. The local Euler-Lagrange equation for this problem is:

$$\frac{d}{dx}(G\delta)(\psi,m) \;\; = \;\; -(\gamma\delta)_x(\psi,m) + (G\delta)_x(\psi,m)\psi'.$$

Since $(G\delta_m)(x,m)=0$, this reduces to

$$\begin{aligned} 0 &= (G_m\delta + \gamma\delta_x + \gamma_x\delta)(\psi,m) \\ &= (m-\psi)\gamma(\psi,m)\frac{\partial}{\partial x}\Big\{\frac{\delta(x,m)}{m-x}\Big\}_{x=\psi}. \end{aligned}$$

This shows formally that the solution of the minimization problem:

$$\min_{\xi<m}\frac{\delta(x,m)}{m-x}$$

provides a solution to the local Euler-Lagrange equation. Finally, the solution of the above minimization problem is known to be given by the right inverse barycenter function b^{-1}, see the proof of Lemma 3.3 in [11].

Proof of Theorem 3.1 To complete the proof of the theorem, it remains to prove that

$$\inf_{\lambda\in\Lambda^\mu}\big\{\mu(\lambda)+u^\lambda(X_0,X_0)\big\} \;\; \ge \;\; \mathbb{E}^{\mathbb{P}_0}_{X_0,X_0}[g\left(X_{\tau^*},X^*_{\tau^*}\right)].$$

To see this, we use the fact that the stopping time τ^* defined in (3.18) is a solution of the Skorohod embedding problem, i.e. $X_{\tau^*}\sim\mu$ and $(X_{t\wedge\tau^*})_{t\ge0}$ is a uniformly integrable martingale, see Azéma and Yor [2, 3]. Moreover $X^*_{\tau^*}$ is integrable. Then, for all $\lambda\in\Lambda^\mu$, it follows from the definition of u^λ that $u^\lambda(X_0,X_0)\ge J(\lambda,\tau^*)$, and therefore:

$$\mu(\lambda)+u^\lambda(X_0,X_0) \;\; \ge \;\; \mu(\lambda)+\mathbb{E}^{\mathbb{P}_0}_{X_0,X_0}\big[g(X_{\tau^*},X^*_{\tau^*})-\lambda(X_{\tau^*})\big] \;\; = \;\; \mathbb{E}^{\mathbb{P}_0}_{X_0,X_0}\big[g(X_{\tau^*},X^*_{\tau^*})\big].$$

4 Multiple-marginals penalized value function

We now return to the multiple-marginal problem of Section 2.3, and we derive the model-free bounds for a lookback derivative security

$$\phi(X^*_{t_n}) \quad \text{given the marginals} \quad X_{t_i}\sim\mu_i \text{ for all } i=1,\dots,n. \tag{4.1}$$

We recall that the probability measures μ_i should be nondecreasing in the convex order or, equivalently, the corresponding European call options prices should satisfy the no-arbitrage condition:

$$c_i \geq c_{i-1} \quad \text{where} \quad c_i(x) := \int_x^\infty (\xi - x)\mu_i(d\xi) \ \text{ for all } \ i = 1, \ldots, n \tag{4.2}$$

and $c_0(x) := (X_0 - x)^+$ corresponds to the Dirac measure $\mu_0 := \delta_{\{X_0\}}$.

We also introduce the barycenter functions corresponding to the probability measures μ_i:

$$b_i(x) \ := \ \frac{\int_{[x,\infty)} y\mu_i(dy)}{\mu([x,\infty))} 1_{\{x \leq r^{\mu^i}\}} + x 1_{\{x \geq r^{\mu^i}\}}. \tag{4.3}$$

Using the notations introduced in Section 2.3, we recall that the model-free upper bound can be expressed in the dual formulation of Proposition 2.1 as:

$$U_n^\mu(\xi) := \inf_{\lambda \in \Lambda_n^\mu} \Big\{ \mu(\lambda) + u^\lambda(X_0, X_0) \Big\}, \quad \text{where} \quad u^\lambda(x,m) := \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}_{x,m}^{\mathbb{P}} \Big[\phi^\lambda(X_t, M_{t_n}) \Big], \tag{4.4}$$

with $\phi^\lambda := \phi - \sum_{i=1}^n \lambda_i$ as in (2.6), and the set of Lagrange multipliers is:

$$\Lambda_n^\mu = \Big\{ \lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \in \mathbb{L}^1(\mu_i) \text{ and } \phi^\lambda\big(X_t, X_{t_n}^*\big)^+ \in \cap_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{L}^1(\mathbb{P}) \Big\}. \tag{4.5}$$

Our approach to solve the present $n-$marginals Skorohod embedding problem is to introduce the sequence of intermediate optimization problems:

$$u_n(x,m) = \phi(m) \quad \text{and} \quad u_{k-1}(x,m) = \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}_{t_{k-1},x,m}^{\mathbb{P}} \Big[u_k^\lambda(X_{t_k}, M_{t_k}) \Big], \quad k \leq n, \tag{4.6}$$

where $\mathbb{E}_{t_{k-1},x,m}^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}\big[. | (X,M)_{t_{k-1}} = (x,m) \big]$, and:

$$u_k^\lambda(x,m) := u_k(x,m) - \lambda_k(x) \quad \text{for} \quad (x,m) \in \mathbf{\Delta}. \tag{4.7}$$

Given this iterative sequence of value functions, it follows from the dynamic programming principle that our problem of interest is given by:

$$u^\lambda = u_0^\lambda \quad \text{for all} \quad \lambda \in \Lambda_n^\mu.$$

From the Dambis-Dubins-Schwartz theorem (see Proposition 3.1 in [11]), we may convert the stochastic control problem in (4.6) into a sequence of optimal stopping problems:

$$u_{k-1}(x,m) \ = \ \sup_{\tau \in \mathcal{T}_\infty} \mathbb{E}_{x,m}^{\mathbb{P}_0} \Big[u_k^\lambda(X_\tau, M_\tau) \Big]. \tag{4.8}$$

Then, denoting by $\mathcal{S}_\infty^n := \{\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{T}_\infty : \tau_1 \leq \cdots \leq \tau_n\}$, we see that

$$U_n^\mu(\xi) = \inf_{\lambda \in \Lambda_n^\mu} \big\{ \mu(\lambda) + u_0^\lambda(X_0, X_0) \big\} \quad \text{where} \quad u_0^\lambda(x,m) := \sup_{\tau \in \mathcal{S}_\infty^n} \mathbb{E}_{x,m}^{\mathbb{P}_0} \big[\phi^\lambda\big(X_\tau, M_{\tau_n}\big) \big], \tag{4.9}$$

and the set Λ_n^μ of (4.5) translates in the present context to:

$$\Lambda_n^\mu = \Big\{ \lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \in \mathbb{L}^1(\mu) \text{ and } \phi^\lambda(X_\tau, X_{\tau_n}^*)^+ \in \mathbb{L}^1(\mathbb{P}_0) \text{ for all } \tau \in \mathcal{S}_\infty^n \Big\}. \tag{4.10}$$

4.1 Main result

The key ingredient for the solution of the present $n-$marginals Skorohod embedding problem turns out to be the following minimization problem:

$$\min_{\zeta_1 \le \ldots \le \zeta_n < m} \sum_{i=1}^{n} \Big(\frac{c_i(\zeta_i)}{m - \zeta_i} - \frac{c_i(\zeta_{i+1})}{m - \zeta_{i+1}} \mathbf{1}_{\{i<n\}} \Big) \quad \text{for all} \quad m \ge 0. \tag{4.11}$$

Set $\zeta^*_{n+1}(m) := m$ for all $m \ge 0$. We shall prove in Lemmas 4.5 and 4.6 below the existence of a minimizer $x \longmapsto \big(\zeta^*_1, \ldots, \zeta^*_n\big)(m)$ defined by

$$\begin{aligned} &\zeta^*_1 = b_1^{-1} \wedge \zeta^*_2, \quad \text{and for} \;\; 1 < i \le n: \\ &\zeta^*_i \in \text{Arg} \min_{0 \le \zeta_i \le \zeta_{i+1}} \Big\{ \frac{c_i(\zeta_i)}{m - \zeta_i} - \Big(\frac{c_{i-1}(\zeta_i)}{m - \zeta_i} - \frac{c_{i-1}(\zeta^*_{i-1})}{m - \zeta^*_{i-1}} \Big) \mathbf{1}_{\{\zeta_i > \zeta^*_{i-1}\}} \Big\}. \end{aligned} \tag{4.12}$$

Our main result relies on the following integrability condition:

$$\int_{X_0}^{\infty} \Big(\frac{c_i\big(\zeta^*_i(m)\big)}{m - \zeta^*_i(m)} + \frac{c_i\big(\zeta^*_{i+1}(m)\big)}{m - \zeta^*_{i+1}(m)} \mathbf{1}_{\{i<n\}} \Big) \phi'(m) dm \;\; < \;\; \infty. \tag{4.13}$$

Theorem 4.1 *Let ϕ be a C^1 nondecreasing Lipschitz function, and assume that the no-arbitrage condition* (4.2) *holds. Then, under the additional integrability condition* (4.13)*:*

$$U^{\mu}_n(\xi) \;\; = \;\; \sum_{i=1}^{n} \int_{X_0}^{\infty} \Big(\frac{c_i(\zeta^*_i(x))}{x - \zeta^*_i(x)} - \frac{c_i(\zeta^*_{i+1}(x))}{x - \zeta^*_{i+1}(x)} \mathbf{1}_{\{i<n\}} \Big) \, \phi'(x) dx.$$

Moreover (λ^*, τ^*) *given in* (4.36) *and* (4.37) *below is a solution of the problem* (4.9)*.*

The proof is reported in the subsequent subsection.

Remark 4.1 (Two-marginals case, [8]) The case $n = 2$ of two given marginals was explicitly solved by Brown, Hobson and Rogers [8]. Our result in Theorem 4.1 agrees with the solution previously obtained by [8].

The following definition was introduced by Madan and Yor [14]. We say that the probability measures $\mu_1, \ldots, \mu_n$ satisfy the *Increasing Mean Residual Value* (IMRV) condition if:

$$b_1 \le \cdots \le b_n. \tag{4.14}$$

We observe that this condition implies that the measures $\mu_1, \ldots, \mu_n$ are nondecreasing in the convex order, i.e. (4.2). Madan and Yor [14] proved provided an extension of the Azéma-Yor solution of the SEP under the IMRV condition. We now snow show that our result agrees with their, and that, in this case, the optimal upper bound only depends on the final marginal μ_n.

Corollary 4.1 *Let ϕ be a C^1 nonincreasing function. Then, under the IMRV condition (4.14), $\zeta_i^* = b_n^{-1}$, $1 \le i \le n$, is a solution of the optimization problem (4.12), and*

$$U_n^\mu(\xi) \;=\; \mu_n^{\mathrm{HL}}(\phi) \;:=\; \int_{X_0}^{\infty} \frac{c_n(b_n^{-1}(x))}{x - b_n^{-1}(x)}\, \phi'(x)dx.$$

Proof To obtain the required result, we shall prove by induction that the solution of the $i-$th optimization problem in (4.12) is given by

$$\zeta_i^* = b_i^{-1} \wedge \zeta_{i+1}^* \quad \text{for all} \quad i = 1, \ldots, n. \tag{4.15}$$

Since $\zeta_{n+1}^*(m) = m$, this would imply that $\zeta_n^* = b_n^{-1}$, and by the IMRV property $\zeta_i^* = b_n^{-1}$ for all $i \le n$. Then the expression of $U_n^\mu(\xi)$ in Theorem 4.1 is a telescopic sum, and we obtain the simple expression stated in the corollary by a direct integration by parts.

It remains to prove (4.15). First, the property holds true for $i = 1$ by (4.12). We next suppose that $\zeta_{i-1}^* = b_{i-1}^{-1} \wedge \zeta_i$, and intend to prove that (4.12) holds true for i. Under the induction hypothesis, the $i-$th optimization problem in (4.12) reduces to:

$$\min\{A_1, A_2\}, \quad A_1 := \min_{\zeta_i \le b_{i-1}^{-1} \wedge \zeta_{i+1}} F_1(\zeta_i), \quad A_2 := \min_{b_{i-1}^{-1} \le \zeta_i \le \zeta_{i+1}} F_2(\zeta_i), \tag{4.16}$$

where we deleted the dependence of b_i in m, and

$$F_1(\zeta_i) := \frac{c_i(\zeta_i)}{m - \zeta_i} \quad \text{and} \quad F_2(\zeta_i) := \frac{c_i(\zeta_i)}{m - \zeta_i} - \left(\frac{c_{i-1}(\zeta_i)}{m - \zeta_i} - \frac{c_i\big(b_{i-1}^{-1}\big)}{m - b_{i-1}^{-1}} \right).$$

(i) Since the function F_1 is nonincreasing to left of $b_i^{-1}(m)$ and nondecreasing to the right of $b_i^{-1}(m)$, it easily seen that $A_1 = F_1\big(b_i^{-1} \wedge \zeta_{i+1}\big)$.

(ii) We next focus on A_2. By direct calculation, we see that

$$\begin{aligned}(m - \zeta_i)^2 F_2'(\zeta_i) &= (b_i(\zeta_i) - m)\mu_i\big([\zeta_i, \infty)\big) - (b_{i-1}(\zeta_i) - m)\mu_{i-1}\big([\zeta_i, \infty)\big) \\ &= c_i(\zeta_i)\frac{b_i(\zeta_i) - m}{b_i(\zeta_i) - \zeta_i} - c_{i-1}(\zeta_i)\frac{b_{i-1}(\zeta_i) - m}{b_{i-1}(\zeta_i) - \zeta_i}\end{aligned}$$

By the IMRV condition, we have $b_i^{-1} \ge b_{i-1}^{-1}$, and therefore:

$$(m - \zeta_i)^2 F_2'(\zeta_i) \;\ge\; (c_i - c_{i-1})(\zeta_i)\frac{b_{i-1}(\zeta_i) - m}{b_{i-1}(\zeta_i) - \zeta_i} \;\ge\; 0,$$

where the last inequality follows from the non-decrease of the μ_i's in the convex order (4.2). Hence F_2 is nondecreasing, and it follows that it attains its minimum at the left boundary $A_2 = F_2\big(b_i^{-1} \wedge \zeta_{i+1}\big)$. In view of (i), this shows that (4.12) holds for ζ_i^*. □

Remark 4.2 An interesting question which we leave for future research is the limit $n \to \infty$ corresponding to continuum of marginals μ_t (which must be in convex order). Such a result would extend the results of Madan and Yor [14] established under the IMRV property.

4.2 Preparation for the upper bound

The function u_{k-1} corresponds to the optimization problem considered in Section 3 with a payoff $g(x,m) = u_k(x,m)$ depending on the spot and the running maximum. This was our original motivation for isolating the one-marginal problem.

To solve the multiple marginals problem, we introduce the iterative sequence of candidate value functions:

$$
\begin{aligned}
v_n(x,m) &:= \phi(m), \quad v_k^\lambda(x,m) := v_k(x,m) - \lambda_k(x), \quad \text{and} && (4.17)\\
v_{k-1}(x,m) &:= v_k^\lambda(x \wedge \psi_k(m), m) + \big(x - \psi_k(m)\big)^+ \partial_x v_k^\lambda\big(\psi_k(m), m\big) && (4.18)\\
&= v_k^\lambda(x,m) - \int_{\psi_k(m)}^{x \vee \psi_k(m)} (x-\xi)\partial_{xx} v_k^\lambda(d\xi, m), && (4.19)
\end{aligned}
$$

where $\psi = (\psi_1, \ldots, \psi_n)$ with ψ_i defined as an arbitrary solution of the ordinary differential equation

$$
-\psi_k' \partial_{xx} v_k^\lambda(\psi_k, m) = \gamma_k(\psi_k, m), \quad \text{with} \quad \gamma_k(x,m) := (m-x)\partial_x \Big\{ \frac{\partial_m v_k(x,m)}{m-x} \Big\}, \qquad (4.20)
$$

which stays strictly below the diagonal. Notice that, in contrast to the one-marginal case, we have dropped here the dependence of v_k in ψ by simply denoting $v_k := v_k^\psi$ and $v_k^\lambda := v_k^{\psi,\lambda}$.

Similar to the one-marginal case, we introduce the weak formulation of this ODE:

$$
\begin{aligned}
&\psi_k(m) < m \ \text{ for all } \ m \geq 0, \ \text{ and}\\
&-\textstyle\int_{\psi(E)} \partial_{xx} v_k^\lambda\big(., \psi_k^{-1}\big)(d\xi) = \int_E \gamma_k(\psi_k, .)(dm) \ \text{ for all } \ E \in \mathcal{B}(\mathbb{R}),
\end{aligned} \qquad (4.21)
$$

and we introduce the set

$$
\Psi_n^\lambda := \Big\{ \psi : \mathbb{R} \to \mathbb{R}^n \text{ with right-continuous entries } \psi_k \text{ satisfying (4.21)}, k \leq n \Big\}. \qquad (4.22)
$$

We also follow the one-marginal case by restricting the minimization in (4.9) to those multipliers λ in the set:

$$
\hat\Lambda_n^\mu \ := \ \Big\{ \lambda \in \Lambda_n^\mu : v_{k-1} \text{ concave in } x \text{ and } v_{k-1} \geq v_k^\lambda \text{ for all } k \leq n \Big\}. \qquad (4.23)
$$

Lemma 4.1 *Let ϕ be a $C^1(\mathbb{R})$ nondecreasing Lipschitz function. Then:*
(i) *for all $i = 1, \ldots, n$, the function v_i satisfies Assumptions A and B, i.e. v_i is C^1 in (x,m), Lipschitz in m uniformly in x, $\partial_{xx} v_i$ exists a.e. and $x \longmapsto \partial_m v_i(x,m)/(m-x)$ is nondecreasing,*
(ii) *for all $i = 1, \ldots, n$, the function $\partial_m v_i$ is concave in x,*
(iii) *$u^\lambda(X_0, X_0) \leq v_0(X_0, X_0)$ for all $\lambda \in \hat\Lambda_n^\mu$ and $\psi \in \Psi_n^\lambda$.*

Proof We first prove (i). First $v_n = \phi$ satisfies Assumptions A and B as it is independent of the $x-$variable, nondecreasing and C^1 Lipschitz. For the remaining cases $i \le n-1$, we proceed by induction, assuming that v_{i+1} satisfies Assumptions A and B, and we intend to show that v_i does as well. We first observe that either one of the following condition is also satisfied by v_{i+1}:

$$v_i(x,m) = \phi(m) \quad \text{nondecreasing, or} \quad \partial_m v_i(m,m) = 0, \tag{4.24}$$

where the first alternative holds for $i = n$. v_{i-1} is clearly C^1, and by using the ODE (4.20) satisfied by v_i, we directly compute that

$$\partial_m v_{i-1}(x,m) = \begin{cases} \partial_m v_i(x,m) & \text{for} \quad x \in (-\infty, \psi_i(m)] \\ \partial_m v_i(\psi_i(m),m)\frac{m-x}{m-\psi_i(m)} & \text{for} \quad x \in [\psi_i(m), m]. \end{cases} \tag{4.25}$$

Then v_{i-1} inherits the Lipschitz property of g in m, uniformly in x. Moreover, $x \longmapsto \partial_m v_{i-1}(x,m)/(m-x)$ is non-decreasing whenever $x \longmapsto \partial_m v_i(x,m)/(m-x)$ is.

We next prove (iii). By the previous step, v_i satisfies Assumptions A and B for all $i = 1,\ldots,n$. Then it follows from Proposition 3.1 that $u_{n-1} \le v_{n-1}$ for all $\psi \in \Psi^{\lambda_n}$. Therefore

$$u_{n-2}(x,m) \quad \le \quad \sup_{\tau_{n-1}\in\mathcal{T}_\infty} \mathbb{E}^{\mathbb{P}}_{x,m}\Big[v^\lambda_{n-1}(X_{\tau_{n-1}}, X^*_{\tau_{n-1}})\Big],$$

and we deduce from a second application of Proposition 3.1 that $u_{n-2} \le v_{n-2}$. The required inequality follows by a backward iteration of this argument.

We finally prove (ii). From (4.25), we see that $\partial_m v_{i-1}$ is concave in x on $\big(-\infty, \psi_i(m)\big)$ and on $\big(\psi_i(m), m\big]$. It remains to verify that $\partial_m v_{i-1}$ is concave at the point $x = \psi_i(m)$. We directly calculate that

$$\partial_{xm} v_{i-1}\big(\psi_i(m)-, m\big) = \partial_{xm} v_i\big(\psi_i(m)-, m\big) \quad \text{and} \quad \partial_{xm} v_{i-1}\big(\psi_i(m)+, m\big) = \frac{-\partial_m v_i\big(\psi_i(m), m\big)}{m-\psi_i(m)}.$$

Then, by the concavity of $\partial_m v_i$ in x, together with (4.24), we have

$$\partial_m v_i\big(\psi_i(m), m\big) + \partial_{xm} v_i\big(\psi_i(m)+, m\big)\big(m-\psi_i(m)\big) \ge \partial_m v_i\big(m,m\big) \ge 0,$$

which implies that $\partial_{xm} v_{i-1}\big(\psi_i(m)-, m\big) \ge \partial_{xm} v_{i-1}\big(\psi_i(m)+, m\big)$. $\square$

Our next result uses the notation:

$$\delta_i(x,m) := c_i(x) - c_0(x)\mathbf{1}_{\{m<X_0\}} \quad (x,m) \in \mathbf{\Delta}. \tag{4.26}$$

Lemma 4.2 *Let ϕ be a C^1 nondecreasing Lipschitz function. Then, for all $\lambda \in \hat{\Lambda}^\mu_n$ and $\psi \in \Psi^\lambda_n$, we have:*

$$\mu(\lambda) + u^\lambda(X_0, X_0) \quad \le \quad \phi(X_0) + \sum_{i=1}^n \int \delta_i\big(\xi, \psi_i^{-1}(\xi)\big)\lambda_i''(d\xi) - \int_{\psi_i(X_0)}^{X_0} c_0(\xi)\partial_{xx} v_i(\xi, X_0)d\xi.$$

Proof This is a direct consequence of Lemma 4.1 obtained by substituting the expression of the v_i's, and using the fact that $\mu_i(\lambda_i) - \lambda_i(X_0) = \int \lambda_i'' d(\mu - \delta_{X_0})$. $\square$

The following result provides the necessary calculations for the terms which appear in Lemma 4.2. We denote:

$$\overline{\psi}_i := \psi_i \wedge \ldots \wedge \psi_n \quad \text{for all} \quad i = 1, \ldots, n, \tag{4.27}$$

and we set $\overline{\psi}_{n+1}(m) := m$, $m \geq 0$.

Lemma 4.3 *For a C^1 nondecreasing Lipschitz function ϕ, $\lambda \in \Lambda_n^\mu$, $\psi \in \Psi_n^\lambda$, and $i \leq n$, we have:*

$$\text{(i)} \int \delta_i\big(., \psi_i^{-1}\big) d\lambda_i' = \int \Big(\frac{\delta_i\big(\overline{\psi}_i, .\big)(m)}{m - \overline{\psi}_i(m)} - \frac{\delta_i\big(\overline{\psi}_{i+1}, \psi_i^{-1} \circ \overline{\psi}_{i+1}\big)(m)}{m - \overline{\psi}_{i+1}(m)} \mathbf{1}_{\{i<n\}} \Big) \Big(\mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_{i+1}\}} \phi' \Big)(m) dm,$$

$$\text{(ii)} \quad \int_{\psi_i(X_0)}^{X_0} c_0(\xi) \partial_{xx} v_i(\xi, X_0) d\xi = -\mathbf{1}_{\{i<n\}} \int_0^{X_0} \frac{c_0\big(\overline{\psi}_{i+1}(m)\big)}{m - \overline{\psi}_{i+1}(m)} \mathbf{1}_{\{\overline{\psi}_{i+1}(m) > \psi_i(X_0)\}} \phi'(m) dm.$$

Proof See Section 5.2. $\square$

Plugging these calculations into the estimate of Lemma 4.2 provides:

Lemma 4.4 *Let ϕ be a C^1 nondecreasing Lipschitz function. Then, for all $\lambda \in \hat{\Lambda}_n^\mu$ and $\psi \in \Psi_n^\lambda$, we have:*

$$\begin{aligned} &\mu(\lambda) + u^\lambda(X_0, X_0) \\ &\qquad \leq \phi(X_0) + \int \sum_{i=1}^n \Big(\frac{\delta_i\big(\overline{\psi}_i(m), m\big)}{m - \overline{\psi}_i(m)} - \frac{\delta_i\big(\overline{\psi}_{i+1}(m), m\big)}{m - \overline{\psi}_{i+1}(m)} \mathbf{1}_{\{i<n\}} \Big) \Big(\phi' \mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_{i+1}\}} \Big)(m) \, dm. \end{aligned}$$

where we recall that $\overline{\psi}_{n+1}(m) := m$, $m \geq 0$.

Proof By Lemmas 4.2 and 4.3 (i), we have

$$\mu(\lambda) + u^\lambda(X_0, X_0) \quad \leq \quad \int \phi'(m) \sum_{i=1}^n \mathbf{1}_{\{\overline{\psi}_i(m) < \overline{\psi}_{i+1}(m)\}} A_i(m) dm,$$

where

$$\begin{aligned} A_i(m) \quad = \quad & \frac{c_i\big(\overline{\psi}_i(m)\big) - c_0\big(\overline{\psi}_i(m)\big) \mathbf{1}_{\{m<X_0\}}}{m - \overline{\psi}_i(m)} \\ & - \frac{c_i\big(\overline{\psi}_{i+1}(m)\big) - c_0\big(\overline{\psi}_{i+1}(m)\big) \big(\mathbf{1}_{\{\overline{\psi}_{i+1}(m) < \psi_i(X_0)\}} + \mathbf{1}_{\{m<X_0\}} \mathbf{1}_{\{\overline{\psi}_{i+1}(m) > \psi_i(X_0)\}} \big)}{m - \overline{\psi}_{i+1}(m)}. \end{aligned}$$

Notice that $m < X_0$ on $\{\overline{\psi}_i(m) < \overline{\psi}_{i+1}(m)\}$. Then

$$\begin{aligned} A_i(m) &= \frac{c_i\big(\overline{\psi}_i(m)\big) - c_0\big(\overline{\psi}_i(m)\big)\mathbf{1}_{\{m<X_0\}}}{m - \overline{\psi}_i(m)} - \frac{c_i\big(\overline{\psi}_{i+1}(m)\big) - c_0\big(\overline{\psi}_{i+1}(m)\big)\mathbf{1}_{\{m<X_0\}}}{m - \overline{\psi}_{i+1}(m)} \\ &\text{on} \quad \{\overline{\psi}_i(m) < \overline{\psi}_{i+1}(m)\}. \end{aligned}$$

□

We now consider the problem of minimization inside the integral in the expression obtained in Lemma 4.4, forgetting about the constraints on the ψ_i's.

Lemma 4.5 *Under the no-arbitrage condition* (4.2), *we have*

$$\min_{\zeta_1 \le \ldots \le \zeta_n < m} \sum_{i=1}^{n} \left\{ \frac{\delta_i(\zeta_i, m)}{m - \zeta_i} - \frac{\delta_i(\zeta_{i+1}, m)}{m - \zeta_{i+1}} \right\} = 0 \quad \textit{for} \quad m < X_0,$$

and the minimum is attained at $\zeta_i^* = 0$, $i = 1, \ldots, n$.

Proof Since $m < X_0$ and $\zeta_i < m$ for all $i \le n$, it follows that $c_0(\zeta_i)\mathbf{1}_{\{m<X_0\}} = c_0(\zeta_i)$. We proceed by induction.
1. Notice that ζ_1 only appears in the first term of the sum. The partial minimization with respect to ζ_1 reduces to

$$\min_{\zeta_1 \le \zeta_2} \frac{c_1(\zeta_1) - c_0(\zeta_1)}{m - \zeta_1}.$$

By the no-arbitrage condition the function to be minimized is nonnegative, and is zero for $\zeta_1^* = 0$.
2. For $2 \le i \le n$, assume that $\zeta_{i-1}^* = 0$ realizes the minimum over ζ_{i-1}. Then, the partial minimization with respect to ζ_i reduces to

$$\min_{0 \le \zeta_i < m} \frac{c_i(\zeta_i) - c_0(\zeta_i)}{m - \zeta_i}\mathbf{1}_{\{\zeta_i < \zeta_{i+1}\}} - \frac{c_{i-1}(\zeta_i) - c_0(\zeta_i)}{m - \zeta_i}.$$

Since $c_i \ge c_{i-1}$ by the no-arbitrage condition, it is clear that the latter minimum is zero and attained at $\zeta_i^* = 0$. □

For our next result, we recall that the barycenter function b_i is nondecreasing from $\mathbb{R}$ to $[X_0, r_i)$ where $r_i \le \infty$ is the upper bound of the support of μ_i. We denote by b_i^{-1} the corresponding right-continuous inverse function, and we recall from Hobson [12] that

$$\frac{c_i(\zeta)}{m - \zeta} \text{ nonincreasing on } \big(-\infty, b_i^{-1}(m)\big], \text{ nondecreasing on } \big[b_i^{-1}(m), \infty\big). \tag{4.28}$$

Lemma 4.6 *Under the no-arbitrage condition (4.2), for $m \geq X_0$, the problem*

$$\min_{\zeta_1 \leq \ldots \leq \zeta_n < m} \sum_{i=1}^{n} \left(\frac{c_i(\zeta_i)}{m - \zeta_i} - \frac{c_i(\zeta_{i+1})}{m - \zeta_{i+1}} \right)$$

has a solution $\zeta_1^*(m) \leq \ldots \leq \zeta_n^*(m) < m$ with:

$$\zeta_1^*(m) = b_1^{-1}(m) \wedge \zeta_2^* \quad \textit{and} \quad \zeta_i^* \quad \textit{nondecreasing for all} \quad i = 1, \ldots, n.$$

Proof We proceed in several steps.

1. We first solve the minimization with respect to ζ_1. Since ζ_1 only appears in the first term of the summation, we are reduced to the problem:

$$\min_{0 \leq \zeta_1 \leq \zeta_2} \frac{c_1(\zeta_1)}{m - \zeta_1}, \tag{4.29}$$

for given $\zeta_2 < m$. We consider two alternative cases:

- if $\zeta_2 > b_1^{-1}(m)$, then it follows from (4.28) that $\zeta_1^*(m) = b_1^{-1}(m)$ is a minimizer of (4.29).
- if $\zeta_2 \leq b_1^{-1}(m)$, then it follows again from (4.28) that the minimizer $\zeta_1^*(m)$ of (4.29) can be chosen to be any value larger than ζ_2; since ζ_1 is restricted to the interval $(-\infty, \zeta_2]$, the only admissible minimizer is $\zeta_1^*(m) = \zeta_2$.

2. For $1 < i \leq n-1$, let ζ_{i-1}^* be the solution from the previous step, and assume that $\zeta_{i-1}^*(m)$ does not depend on ζ_i on $\{\zeta_{i-1}^*(m) < \zeta_i\}$ for all $m \geq 0$. Then, the partial minimization with respect to ζ_i for arbitrary ζ_{i+1} leads to the problem:

$$\min_{\zeta_i \leq \zeta_{i+1}} \left(\frac{c_{i-1}\big(\zeta_{i-1}^*(m)\big)}{m - \zeta_{i-1}^*(m)} - \frac{c_{i-1}\big(\zeta_i\big)}{m - \zeta_i} \right) \mathbf{1}_{\left\{\zeta_{i-1}^*(m) < \zeta_i\right\}} + \left(\frac{c_i\big(\zeta_i\big)}{m - \zeta_i} - \frac{c_i\big(\zeta_{i+1}\big)}{m - \zeta_{i+1}} \right) \mathbf{1}_{\left\{\zeta_i < \zeta_{i+1}\right\}}$$

which reduces to

$$M_i \quad := \quad \min_{0 \leq \zeta_i \leq \zeta_{i+1}} \quad \frac{c_i\big(\zeta_i\big)}{m - \zeta_i} - \left(\frac{c_{i-1}\big(\zeta_i\big)}{m - \zeta_i} - \frac{c_{i-1}\big(\zeta_{i-1}^*(m)\big)}{m - \zeta_{i-1}^*(m)} \right) \mathbf{1}_{\{\zeta_i > \zeta_{i-1}^*(m)\}}. \tag{4.30}$$

A similar optimization problem to the latter is analyzed in Lemma 3.2 of Brown, Hobson and Rogers [8]. The difference between our context and their is the presence of the restriction $\zeta_i \leq \zeta_{i+1}$ and the fact that, in [8], $\zeta_{i-1}^* = b_{i-1}^{-1}$ is the Azéma-Yor solution. For this reason, we can not simply refer to their result, and we analyze the above optimization problem in the next step.

3. By the continuity of the objective function in (4.30), existence holds for this optimization. However, as observed by [8], there is no guarantee of uniqueness. We shall denote by $\zeta_i^*(m)$ an arbitrary solution of (4.30), i.e.

$$\zeta_i^*(m) \in \big(-\infty, \zeta_{i+1}\big] \quad \text{and} \quad M_i = \frac{c_i\big(\zeta_i^*(m)\big)}{m - \zeta_i^*(m)} - \left(\frac{c_{i-1}\big(\zeta_i^*(m)\big)}{m - \zeta_i^*(m)} - \frac{c_{i-1}\big(\zeta_{i-1}^*(m)\big)}{m - \zeta_{i-1}^*(m)} \right) \mathbf{1}_{\{\zeta_i^*(m) > \zeta_{i-1}^*(m)\}}$$

4. In this step, we fix $i = 2, \ldots, n$, and we prove the following analogue of Lemma 3.2 in [8]:

$$c_i \geq c_{i-1} \quad \text{implies that} \quad m \longmapsto \zeta_i^*(m) \;\; \text{is nondecreasing.} \tag{4.31}$$

Notice that the subsequent proof is based on different arguments than those of [8]. For notational simplicity, we set $i = 2$ and we drop the dependence of ζ_1^* on m. We first decompose the minimization problem (4.30) as:

$$\begin{aligned} M_2 &= \min\left\{ \min_{0\leq\zeta_2\leq\zeta_1^*\wedge\zeta_3} \frac{c_2\big(\zeta_2\big)}{m-\zeta_2}\,,\; \frac{c_1\big(\zeta_1^*\big)}{m-\zeta_1^*} + \min_{\zeta_1^*\wedge\zeta_3\leq\zeta_2\leq\zeta_3} \frac{c_2(\zeta_2)-c_1(\zeta_1)}{m-\zeta_1} \right\} \\ &= \min\left\{ \frac{c_2\big(b_2^{-1}\wedge\zeta_1^*\wedge\zeta_3\big)}{m-b_2^{-1}\wedge\zeta_1^*\wedge\zeta_3}\,,\; \frac{c_1\big(\zeta_1^*\big)}{m-\zeta_1^*} + \min_{\zeta_1^*\wedge\zeta_3\leq\zeta_2\leq\zeta_3} \frac{c_2(\zeta_2)-c_1(\zeta_1)}{m-\zeta_1} \right\}, \end{aligned}$$

where we have used (4.28). We next concentrate on the minimization problem

$$\min_{\zeta_1^*\wedge\zeta_3\leq\zeta_2\leq\zeta_3} H\big(\zeta_2\big) \quad \text{where} \quad H(\zeta) := \frac{c_2(\zeta)-c_1(\zeta)}{m-\zeta}\,.$$

On the interval $\big[\zeta_1^*\wedge\zeta_3, \zeta_3\big]$, the function H attains a minimum either on the boundaries, or at an interior critical point. We denote by $\mathcal{H}_{\min}$ the set of interior minimizers of H on $\big[\zeta_1^*\wedge\zeta_3, \zeta_3\big]$. Then:

$$M_2 \;=\; \min\left\{ \frac{c_2\big(b_2^{-1}\wedge\zeta_1^*\wedge\zeta_3\big)}{m-b_2^{-1}\wedge\zeta_1^*\wedge\zeta_3}\,,\; H(\zeta_3)\,,\; \min_{\zeta_2\in\mathcal{H}_{\min}} H\big(\zeta_2\big) \right\}. \tag{4.32}$$

By direct calculation, we see that the local minimizers of H satisfy the first and second order conditions of optimality:

$$m = \frac{\int_\zeta^\infty y\mu_2(dy) - \int_\zeta^\infty y\mu_1(dy)}{\int_\zeta^\infty \mu_2(dy) - \int_\zeta^\infty \mu_1(dy)} \quad \text{and} \quad H''(\zeta) = \frac{f_2(\zeta)-f_1(\zeta)}{m-\zeta} \geq 0 \;\; \text{for all} \;\; \zeta\in\mathcal{H}_{\min}, \tag{4.33}$$

where $f_i(\zeta) := \partial_\zeta \int_0^\zeta \mu_i(dy)$ in the distribution sense. Moreover, since the elements of $\mathcal{H}_{\min}$ are less than m, we have:

$$m-\zeta = \frac{c_2(\zeta)-c_1(\zeta)}{\int_\zeta^\infty \mu_2(dy) - \int_\zeta^\infty \mu_1(dy)} \;\geq 0 \;\; \text{and then} \;\; \int_\zeta^\infty \mu_2(dy) \geq \int_\zeta^\infty \mu_1(dy) \tag{4.34}$$

by the no-arbitrage condition $c_2 \geq c_1$. We now differentiate the first order optimality condition in (4.33) to see that:

$$(m-\zeta)\big(f_2(\zeta)-f_1(\zeta)\big)\zeta' \;=\; \int_\zeta^\infty \mu_2(dy) - \int_\zeta^\infty \mu_1(dy) \;\geq\; 0$$

by (4.34). Here again, the derivatives must be understood in the distribution sense. By the second order condition in (4.33), this proves that any local minimizer in $\mathcal{H}_{\min}$ is nondecreasing. Returning to our minimization problem (4.32), we see that the minimizer $b_2^{-1}\wedge\zeta_1^*\wedge\zeta_3$ in the first argument is also nondecreasing, and the second argument corresponds to a constant minimizer. Hence the minimizer of the problem m_2 is nonincreasing. □

4.3 Proof Theorem 4.1

1. Given the results of Lemma 4.4, we prove in this first step that the pointwise minimization of Lemmas 4.5 and 4.6 can be achieved by some vector of Lagrange multipliers $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*) \in \Lambda_n^\mu$, thus implying that our required upper bound satisfies:

$$U_n^\mu(\xi) \le \phi(X_0) + \int \sum_{i=1}^n \Big(\frac{\delta_i\big(\zeta_i^*(m), m\big)}{m - \zeta_i^*(m)} - \frac{\delta_i\big(\zeta_{i+1}^*(m), m\big)}{m - \zeta_{i+1}^*(m)} \mathbf{1}_{\{i<n\}} \Big) \Big(\phi' \mathbf{1}_{\{\zeta_i^* < \zeta_{i+1}^*\}} \Big)(m)\, dm. \quad (4.35)$$

In order to define λ^*, we first introduce the family of nondecreasing functions ψ_i^* satisfying:

$$\psi_1^* := b_1^{-1}, \quad \psi_n^* := \zeta_n^*, \quad \text{and} \quad \overline{\psi}_i^* := \psi_i^* \wedge \cdots \wedge \psi_n^* = \zeta_i^*, \quad 1 < i < n,$$

i.e. ψ_i^* is a nondecreasing extension of ζ_i^* for all $i = 1, \dots, n$. Then the following Lagrange multipliers are obtained by plugging the ψ_i^*'s in (4.20):

$$\lambda_k^*(x) \quad := \quad \int_{\ell^\mu}^x \Big(\partial_x v_k\big(y, \psi_k^{-1}(y)\big) + \int_{\ell^\mu}^y \frac{\partial_m v_k\big(\xi, \psi_k^{-1}(\xi)\big)}{\psi_k^{-1}(\xi) - \xi} d\psi_k^{-1}(\xi) \Big) dy. \quad (4.36)$$

1.1. Recall from Lemma 4.1 (ii) that $\partial_m v_k$ in x. Then, proceeding exactly as in Step 3 of the proof of Proposition 3.2, it follows that:

$$\begin{aligned} v_k^{\lambda^*}(x,m) &\le v_k(x,m) - v_k(x, X_0) + v_k(0, X_0) - \int_{\ell^\mu}^x \int_{\ell^\mu}^y \gamma_k\big(\xi, \psi_k^{-1}(\xi)\big) d\psi_k^{-1}(\xi) dy \\ &\le C|m - X_0| + v_k(0, X_0), \end{aligned}$$

by the Lipschitz property of v_k in m, uniformly in x, see Lemma 4.1 (i). This implies that $v_k^{\lambda^*}(X_\tau, M_\tau)^+$ is integrable.

1.2. The final ingredient to verify, in order for $\lambda^* \in \Lambda_n^\mu$ which implies that inequality (4.35) holds, is that $\lambda_i^* \in \mathbb{L}^1(\mu_i)$. To see this, we follow the same calculations as in the proof of Lemma 4.4 to see that

$$\mu_i(\lambda_i^*) \quad \le \quad \text{Const} + \int \Big(\frac{c_i\big(\psi_i^*(m)\big)}{m - \psi_i^*(m)} - \frac{c_i\big(\overline{\psi}_{i+1}^*(m)\big)}{m - \overline{\psi}_{i+1}^*(m)} \Big) \phi'(m) \mathbf{1}_{\{\psi_i^*(m) < \overline{\psi}_{i+1}^*(m)\}} dm$$

proving the required integrability by our condition (4.13).

2. We now prove that equality holds in (4.35). To do this, we follow the Brown-Hobson-Rogers construction reported in Section 3.3 of [8], we define stopping times τ_1^* and τ_2^* such that $X_{\tau_1^*} \sim \mu_1$, $X_{\tau_2^*} \sim \mu_2$, and $\{B_{t \wedge \tau_2^*}, t \ge 0\}$ is a uniformly integrable martingale. This construction can obviously be iterated so as to define stopping times $(\tau_1^*, \dots, \tau_n^*) \in \mathcal{S}_\infty^n$ such that $X_{\tau_i^*} \sim \mu_i$. The resulting stopping times are given by

$$\tau_1^* := \mathbf{T}_1^{(1)}(0) \quad \text{and} \quad \tau_k^* := \mathbf{T}_k\big(\tau_{k-1}^*\big), \quad k \le n,$$

where

$$\mathbf{T}_k(\tau) := \begin{cases} \mathbf{T}_k^{(1)}(\tau) := \inf\big\{t > \tau : X_t \le \psi_k^*(X_t^*)\big\}, & \text{if } \; X_\tau > \psi_k^*(X_\tau^*) \\ \mathbf{T}_k^{(2)}(\tau) = \tau' \;\text{ s.t. }\; X_{\tau'}^* = X_\tau^*, \;\; X_{\tau'} \sim \frac{\mu_k \mathbf{1}_{(\psi_k^*(X_\tau^*-),\psi_k^*(X_\tau^*))}}{\mu_k\{(\psi_k^*(X_\tau^*-),\psi_k^*(X_\tau^*))\}}, & \text{if } \; \Delta\psi_k^*(X_\tau^*) > 0 \\ \mathbf{T}_k^{(3)}(\tau) = \tau, \;\; \text{if } \; \Delta\psi_k^*(X_\tau^*) = 0. \end{cases} \tag{4.37}$$

where $\Delta\psi_k^*(X_\tau^*) := \psi_k^*(X_\tau^*) - \psi_k^*(X_\tau^*-)$, and the existence of τ' with the definition of $\mathbf{T}_k^{(2)}$ is proved in [8]. Finally, by the expression of $U_n^\mu(\xi)$ in (4.9), it follows that:

$$U_n^\mu(\xi) \;\; \ge \;\; \inf_{\lambda\in\Lambda_n^\mu} \Big\{\mu(\lambda) + \mathbb{E}^{\mathbb{P}_0}\Big[\phi\big(X_{\tau_n^*}^*\big) - \sum_{i=1}^n \lambda_i\big(X_{\tau_i^*}\big)\Big]\Big\} \; = \; \mathbb{E}^{\mathbb{P}_0}\big[\phi\big(X_{\tau_n^*}^*\big)\big].$$

□

5 Appendix

5.1 Proof of Theorem 2.1

We proceed in two steps.

1. Let Y_0 be so that $Y_T^H \ge \xi$ for some $H \in \mathcal{H}$. Then, it follows from (2.3) that $Y_0 \ge \mathbb{E}^{\mathbb{P}}[\xi]$ for all $\mathbb{P} \in \mathcal{P}_\infty$. This shows that $U^0(\xi) \ge \sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi]$.

2. To prove the reverse inequality, it suffices to consider the case $\xi \in \mathbb{L}^1(\mathbb{P})$ for all $\mathbb{P}$ and $\sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi] < \infty$.

Define $G_n := G$, and for $1 \le k \le n$:

$$w_{k-1}(x_1,\ldots,x_{k-1},y,m) \;\; := \;\; \sup_{\mathbb{P}\in\mathcal{P}_+^\infty} \mathbb{E}^{\mathbb{P}}_{t_{k-1},y,m}\big[G_k\big(x_1,\ldots,x_{k-1},X_{t_k},M_{t_k}\big)\big] \tag{5.1}$$

$$G_{k-1}(x_1,\ldots,x_{k-1},m) \;\; := \;\; w_{k-1}(x_1,\ldots,x_{k-1},x_{k-1},m). \tag{5.2}$$

We next consider the process:

$$V_t := \sum_{i=1}^n w_{k-1}\big(X_{t_1},\ldots,X_{t_{k-1}},X_t,X_t^*\big)\mathbf{1}_{[t_{k-1},t_k)}(t) + w_{n-1}\big(X_{t_1},\ldots,X_{t_n},X_{t_n}^*\big)\mathbf{1}_{\{t_n\}}(t), \tag{5.3}$$

for $t \in [0,t_n]$. By the dynamic programming principle, we see immediately that the process $\{V_t, t\in[t_{k-1},t_k]\}$ is a $\mathbb{P}-$supermartingale for all $\mathbb{P}\in\mathcal{P}_\infty$, and that w_{k-1} is concave in the $y-$variable for all $k=1,\ldots,n$. In particular w_{k-1} has left and right derivatives at any point $y\in\mathbb{R}$. Then, by the Tanaka formula, we may conclude that:

$$V_t \le V_0 + \int_0^t \tilde{H}_s dX_s \quad \text{where} \quad \tilde{H}_s := \partial_y^+ w_{k-1}\big(X_{t_1},\ldots,X_{t_{k-1}},X_t,X_t^*\big)\mathbf{1}_{[t_{k-1},t_k)}(t),$$

where ∂_y^+ denotes the right-derivative operator. Set $Y_0 := V_0$, Then, the previous inequality together with the definition of V imply that

$$Y_t^{\tilde{H}} \geq \mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t] \quad \text{for all } \ \mathbb{P} \in \mathcal{P}_\infty, \quad \text{and} \quad Y_{t_n}^{\tilde{H}} \geq V_{t_n} \geq \xi, \quad \mathcal{P}_\infty - \text{q.s.}$$

We may then conclude that $\tilde{H} \in \mathcal{H}$, and $Y_0 = V_0 \geq U^0(\xi)$ by the definition of the super-hedging problem $U^0(\xi)$. As a by-product, we obtain that $\tilde{H}$ achieves the maximum in the definition of $U^0(\xi)$.

5.2 Proof of Lemma 4.3

We start with the computation of $\gamma_i(\psi_i, .)$, as defined in (4.20), in terms of g and the ψ_i's.

Lemma 5.1 *For all $i < n$, we have* $\gamma_i\big(\psi_i(m), m\big) = \frac{\phi'(m)}{m-\psi_i(m)} \mathbf{1}_{\{\psi_i < \overline{\psi}_{i+1}\}}$.

Proof By direct differentiation of (4.17), we see that:

$$\begin{aligned}
\partial_m v_{i-1}(x,m) &= \partial_m v_i(x \wedge \psi_i(m), m) \\
&\quad + (x-\psi_i(m))^+ \big[\partial_{xx} v_i(\psi_i(m), m)\psi_i'(m) + \partial_{xm} v_i(\psi_i(m), m)\big].
\end{aligned}$$

Using the ODE satisfied by ψ_i, this provides:

$$\begin{aligned}
\partial_m v_{i-1}(x,m) &= \partial_m v_i(x \wedge \psi_i(m), m) + \frac{(x-\psi_i(m))^+}{m-\psi_i(m)} \partial_m v_i(x \wedge \psi_i(m), m) \\
&= \frac{m - x \vee \psi_i(m)}{m-\psi_i(m)} \partial_m v_i(x \wedge \psi_i(m), m). \qquad (5.4)
\end{aligned}$$

Differentiating this expression with respect to x, we also compute that:

$$\begin{aligned}
\partial_{mx} v_{i-1}(x,m) &= \mathbf{1}_{\{x<\psi_i(m)\}} \partial_{mx} v_i(x \wedge \psi_i(m), m) \qquad (5.5) \\
&\quad + \mathbf{1}_{\{x>\psi_i(m)\}} \frac{-1}{m-\psi_i(m)} \partial_m v_i(x \wedge \psi_i(m), m).
\end{aligned}$$

From the expression of γ_i, it follows from (5.4) and (5.5) that:

$$\gamma_{i-1}(x,m) = \mathbf{1}_{\{x<\psi_i(m)\}} \gamma_i(x,m) = \cdots = \mathbf{1}_{\{x<\overline{\psi}_i(m)\}} \gamma_n(x,m) = \mathbf{1}_{\{x<\overline{\psi}_i(m)\}} \frac{\phi'(m)}{m-x}.$$

□

Proof of Lemma 4.3 (i) For any integrable function φ, the following claim:

$$\begin{aligned}
\int \varphi(\xi) \lambda_i''(\xi) d\xi &= \int \Big(\frac{\varphi(\psi_i(m))}{m-\psi_i(m)} \mathbf{1}_{\{\psi_i(m) < \overline{\psi}_{i+1}(m)\}} - \sum_{j=i+1}^{k} \frac{\varphi(\psi_j(m))}{m-\psi_j(m)} \mathbf{1}_{\{\psi_i(m) < \psi_j(m) = \overline{\psi}_j(m)\}} \Big) \phi'(m) dm \\
&\quad + \int \varphi(\xi) \Big[\partial_{xx} v_k\big(\xi, \psi_i^{-1}(\xi)\big) - \partial_{xx} v_k\big(\xi, (\psi_{i+1}^{-1} \vee \ldots \vee \psi_k^{-1})(\xi)\big) \Big] \qquad (5.6) \\
&\qquad \mathbf{1}_{\{\psi_i^{-1}(\xi) > (\psi_{i+1}^{-1} \vee \ldots \vee \psi_k^{-1})(\xi)\}} d\xi,
\end{aligned}$$

which will be proved below by induction, implies the required result by applying it to the function $\varphi(\xi) = \delta_i(\xi, \psi_i^{-1}(\xi))$, with $k = n-1$, and using the fact that $v_n = \phi$ is independent of x.

We next start verifying (5.6) for $k = i+1$. The first ingredient for the verification of (5.6) is the fact that

$$\partial_{xx} v_j(x,m) = \partial_{xx} v_{j+1}^\lambda(x,m) \mathbf{1}_{\{x<\psi_{j+1}(m)\}}, \quad \text{where} \quad v_j^\lambda = v_j - \lambda_j. \tag{5.7}$$

which can be immediately checked from the expression of v_i in (4.17).

1. To see that (5.6) holds true with $k = i+1$, we first decompose the integral so as to use the ODE satisfied by ψ_i:

$$\begin{aligned} \int \varphi \lambda_i'' &= -\int \varphi(\xi) \partial_{xx} v_i^\lambda\big(\xi, \psi_i^{-1}(\xi)\big) d\xi + \int \varphi(\xi) \partial_{xx} v_i\big(\xi, \psi_i^{-1}(\xi)\big) d\xi \\ &= \int \varphi\big(\psi_i(m)\big) \gamma_i\big(\psi_i(m), m\big) dm + \int \varphi(\xi) \partial_{xx} v_i\big(\xi, \psi_i^{-1}(\xi)\big) d\xi. \end{aligned}$$

We next substitute the expression of $\gamma_i(\psi_i, .)$ from Lemma 5.1, and use (5.7) for the second integral:

$$\begin{aligned} \int \varphi \lambda_i'' &= \int \frac{\varphi(\psi_i(m))}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i(m)<\overline{\psi}_{i+1}(m)\}} dm + \int \varphi(\xi) \partial_{xx} v_{i+1}^\lambda\big(\xi, \psi_i^{-1}(\xi)\big) \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}} d\xi \\ &= \int \frac{\varphi(\psi_i(m))}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i(m)<\overline{\psi}_{i+1}(m)\}} dm + \int \varphi(\xi) \partial_{xx} v_{i+1}^\lambda\big(\xi, \psi_{i+1}^{-1}(\xi)\big) \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}} d\xi \\ &\quad + \int \varphi(\xi) \big[\partial_{xx} v_{i+1}^\lambda\big(\xi, \psi_i^{-1}(\xi)\big) - \partial_{xx} v_{i+1}^\lambda\big(\xi, \psi_{i+1}^{-1}(\xi)\big)\big] \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}} d\xi. \end{aligned}$$

Then, by using again the ODE (3.12) satisfied by ψ_{i+1} together with the expression of $\gamma_{i+1}(\psi_{i+1}, .)$ from Lemma 5.1, we get:

$$\begin{aligned} \int \varphi \lambda_i'' &= \int \frac{\varphi\big(\psi_i(m)\big)}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i(m)<\overline{\psi}_{i+1}(m)\}} dm - \int \frac{\varphi\big(\psi_{i+1}(m)\big)}{m - \psi_{i+1}(m)} \mathbf{1}_{\{\psi_i(m)<\psi_{i+1}(m)=\overline{\psi}_{i+1}(m)\}} d\xi \\ &\quad + \int \varphi(\xi) \big[\partial_{xx} v_{i+1}^\lambda\big(\xi, \psi_i^{-1}(\xi)\big) - \partial_{xx} v_{i+1}^\lambda\big(\xi, \psi_{i+1}^{-1}(\xi)\big)\big] \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}} d\xi, \end{aligned}$$

which we recognize to be the required equality (5.6) for $k = i+1$.

2. We next assume that (5.6) holds for some $k < n-1$, and verify it for $k+1$. For simplicity,

we denote $\psi^{-1}_{i+1,j} := \psi^{-1}_{i+1} \vee \cdots \vee \psi^{-1}_{j}$. By (5.7), we compute that:

$$\begin{aligned}
A &:= \int \varphi(\xi)\Big[\partial_{xx}v_k\big(\xi,\psi_i^{-1}(\xi)\big) - \partial_{xx}v_k\big(\xi,\psi^{-1}_{i+1,k}(\xi)\big)\Big]\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi^{-1}_{i+1,k}(\xi)\}}d\xi \\
&= \int \varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi^{-1}_{i+1,k}(\xi)\}}\Big[\big\{\partial_{xx}v_{k+1}\big(\xi,\psi_i^{-1}(\xi)\big) - \lambda''_{k+1}(\xi)\big\}\mathbf{1}_{\{\psi^{-1}_{k+1}(\xi)<\psi_i^{-1}(\xi)\}} \\
&\qquad\qquad -\big\{\partial_{xx}v_{k+1}\big(\xi,\psi^{-1}_{i+1,k}(\xi)\big) - \lambda''_{k+1}(\xi)\big\}\mathbf{1}_{\{\psi^{-1}_{k+1}(\xi)<\psi^{-1}_{i+1,k}(\xi)\}}\Big]d\xi \\
&= \int \varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi^{-1}_{i+1,k}(\xi)\}}\Big[\mathbf{1}_{\{\psi^{-1}_{i+1,k}(\xi)<\psi^{-1}_{k+1}(\xi)<\psi_i^{-1}(\xi)\}}\partial_{xx}v^{\lambda}_{k+1}\big(\xi,\psi_i^{-1}(\xi)\big) \\
&\qquad\qquad +\mathbf{1}_{\{\psi^{-1}_{k+1}(\xi)<\psi^{-1}_{i+1,k}(\xi)\}}\big\{\partial_{xx}v_{k+1}\big(\xi,\psi_i^{-1}(\xi)\big) - \partial_{xx}v_{k+1}\big(\xi,\psi^{-1}_{i+1,k}(\xi)\big)\big\}\Big]d\xi \\
&= \int \varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi^{-1}_{i+1,k}(\xi)\}}\Big[\mathbf{1}_{\{\psi^{-1}_{i+1,k}(\xi)<\psi^{-1}_{k+1}(\xi)<\psi_i^{-1}(\xi)\}}\partial_{xx}v^{\lambda}_{k+1}\big(\xi,\psi^{-1}_{k+1}(\xi)\big) \\
&\qquad\qquad +\mathbf{1}_{\{\psi^{-1}_{i+1,k}(\xi)<\psi^{-1}_{k+1}(\xi)<\psi_i^{-1}(\xi)\}}\big\{\partial_{xx}v^{\lambda}_{k+1}\big(\xi,\psi_i^{-1}(\xi)\big) - \partial_{xx}v^{\lambda}_{k+1}\big(\xi,\psi^{-1}_{k+1}(\xi)\big)\big\} \\
&\qquad\qquad +\mathbf{1}_{\{\psi^{-1}_{k+1}(\xi)<\psi^{-1}_{i+1,k}(\xi)\}}\big\{\partial_{xx}v_{k+1}\big(\xi,\psi_i^{-1}(\xi)\big) - \partial_{xx}v_{k+1}\big(\xi,\psi^{-1}_{i+1,k}(\xi)\big)\big\}\Big]d\xi
\end{aligned}$$

Putting together the two last terms, we see that:

$$\begin{aligned}
A &= \int \varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi^{-1}_{i+1,k}(\xi)\}}\Big[\mathbf{1}_{\{\psi^{-1}_{i+1,k}(\xi)<\psi^{-1}_{k+1}(\xi)<\psi_i^{-1}(\xi)\}}\partial_{xx}v^{\lambda}_{k+1}\big(\xi,\psi^{-1}_{k+1}(\xi)\big) \\
&\qquad\qquad +\mathbf{1}_{\{\psi^{-1}_{k+1}(\xi)<\psi_i^{-1}(\xi)\}}\big\{\partial_{xx}v^{\lambda}_{k+1}\big(\xi,\psi_i^{-1}(\xi)\big) - \partial_{xx}v^{\lambda}_{k+1}\big(\xi,\psi^{-1}_{i+1,k+1}(\xi)\big)\big\}\Big]d\xi
\end{aligned}$$

Finally, using the ODE (3.12) satisfied by ψ_{k+1} in the first term, together with the expression of $\gamma_{k+1}(\psi_{k+1},.)$ from Lemma 5.1, we see that

$$\begin{aligned}
A &= -\int \varphi\big(\psi_{k+1}(m)\big)\frac{\varphi\big(\psi_{k+1}(m)\big)}{\psi_{k+1}(m)-m}\mathbf{1}_{\{\psi_i(m)<\psi_{k+1}(m)=\overline{\psi}_{k+1}(m)\}}dm \\
&\quad + \int \varphi(\xi)\big[\partial_{xx}v^{\lambda}_{k+2}\big(\xi,\psi_i^{-1}(\xi)\big) - \partial_{xx}v^{\lambda}_{k+2}\big(\xi,\psi^{-1}_{i+1,k+2}(\xi)\big)\big]\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi^{-1}_{i+1,k+1}(\xi)\}}d\xi,
\end{aligned}$$

which is precisely the required expression in order to justify that (5.6) holds for $k+1$. □

Proof of Lemma 4.3 (ii) By an induction argument on the line of the previous proof of item (i), we see that:

$$\begin{aligned}
\int_{\psi_i(X_0)}^{X_0} c_0\partial_{xx}v_i(.,X_0) = &-\sum_{j=i+1}^{k}\int_0^{X_0}\frac{c_0\big(\psi_j(m)\big)}{m-\psi_j(m)}\mathbf{1}_{\{\psi_i(X_0)<\psi_j(m)=\overline{\psi}_{i+1}(m)\}}\phi'(m)dm \\
&+\int c_0(\xi)\mathbf{1}_{\{\psi^{-1}_{i+1,k}(\xi)<X_0<\psi_i^{-1}(\xi)\}}\big[\partial_{xx}v_k(\xi,X_0) - \partial_{xx}v_k(\xi,\psi^{-1}_{i+1,k}(\xi))\big], \qquad (5.8)
\end{aligned}$$

where we denoted, as in the previous proof, $\psi^{-1}_{j,k} := \psi^{-1}_{j} \vee \cdots \vee \psi^{-1}_{k}$ for $j \le k$. The required result follows by taking $k = n$ in (5.8). □

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References

[1] Acciaio, B., Beiglböck, F., Penkner, F., Schachermayer, W. and Temme, J. (2012), *Trajectorial Interpretation of Doob's martingale inequalities.* Preprint arXiv:1202.0447v1 .

[2] Azéma, J. and Yor, M. (1979), *Une solution simple au problème de Skorokhod*, Séminaire de Probabilités XIII: 06, 90-115, LNM 721.

[3] Azéma, J. and Yor, M. (1979), *Le problème de Skorokhod: complément à l'exposé précédent*, Séminaire de Probabilités XIII: 06, 625-633, LNM 721.

[4] Beiglböck, M., Henry-Labordère, P. and Penkner, F. (2011), *Model independent Bounds for Option Prices: A Mass Transport Approach.* Preprint, arXiv:1106.5929.

[5] Benamou, J. D. and Brenier, Y. (2000), *A computational fluid mechanics solution to the Monge Kantorovich mass transfer problem*, Numer. Math., 84, 375393.

[6] Bonnans, F. and Tan, X. (2011), *A model-free no-arbitrage price bound for variance options*, preprint.

[7] Breeden, D.T. and Litzenberger, R.H. (1978), *Prices of state-contingent claims implicit in option prices*, J. Business, 51, 621651.

[8] Brown, H., Hobson, D. and Rogers, L.C.G. (1998), *The maximum maximum of a martingale constrained by an intermediate law*, Séminaire de Probabilités XXXII, pp250-263.

[9] Cox, A. and Oblój, J. (2011), *Robust pricing and hedging of double no-touch options*, Finance and Stochastics 15(3), 573-605.

[10] Denis, L. and Martini, C. (2006), *A Theorectical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty*, Annals of Applied Probability 16, 2, 827-852.

[11] Galichon, A., Henry-Labordère, P., Touzi, N., *A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options*, submitted (2011).

[12] Hobson, D. (1998), *Robust hedging of the lookback option*, Finance and Stochastics, 2, pp329-347, 1998.

[13] Hobson, D. and Klimmek, M. (2010), *Maximising functionals of the joint law of the maximum and terminal value in the Skorokhod embedding problem*, arXiv:1012.3909.

[14] Madan, D.B, Yor, M. (2002), *Making Markov martingales meet marginals: with explicit constructions*, Bernoulli 8(4), 509536.

[15] Peng, S. (2007), *G-Brownian motion and dynamic risk measure under volatility uncertainty*, arXiv:0711.2834v1.

[16] Soner, H. M., Touzi, N. and Zhang, J. (2009), *Quasi-sure Stochastic Analysis through Aggregation*, arXiv:1003.4431v1. To appear in *Electronic Journal of Probability*.

[17] Tan, X. and Touzi, N (2011), *Dual Formulation of the optimal transportation problem along controlled stochastic dynamics*. Preprint.